

## NAG Toolbox

### nag\_pde\_1d\_parab\_remesh\_fd (d03pp)

#### 1 Purpose

`nag_pde_1d_parab_remesh_fd` (d03pp) integrates a system of linear or nonlinear parabolic partial differential equations (PDEs) in one space variable, with scope for coupled ordinary differential equations (ODEs), and automatic adaptive spatial remeshing. The spatial discretization is performed using finite differences, and the method of lines is employed to reduce the PDEs to a system of ODEs. The resulting system is solved using a Backward Differentiation Formula (BDF) method or a Theta method (switching between Newton's method and functional iteration).

#### 2 Syntax

```
[ts, u, x, rsave, isave, ind, user, cwsav, lwsav, iwsav, rwsav, ifail] =
nag_pde_1d_parab_remesh_fd(npde, m, ts, tout, pdedef, bndary, uvinit, u, x,
ncode, odedef, xi, rtol, atol, itol, norm_p, laopt, algopt, remesh, xfix,
nrmesh, dxmesh, trmesh, ipminf, monitf, rsave, isave, itask, itrace, ind,
cwsav, lwsav, iwsav, rwsav, 'npts', npts, 'nxi', nxi, 'neqn', neqn, 'nxfix',
nxfix, 'xratio', xratio, 'con', con, 'user', user)

[ts, u, x, rsave, isave, ind, user, cwsav, lwsav, iwsav, rwsav, ifail] = d03pp
(npde, m, ts, tout, pdedef, bndary, uvinit, u, x, ncode, odedef, xi, rtol,
atol, itol, norm_p, laopt, algopt, remesh, xfix, nrmesh, dxmesh, trmesh,
ipminf, monitf, rsave, isave, itask, itrace, ind, cwsav, lwsav, iwsav, rwsav,
'npts', npts, 'nxi', nxi, 'neqn', neqn, 'nxfix', nxfix, 'xratio', xratio, 'con',
con, 'user', user)
```

**Note:** the interface to this routine has changed since earlier releases of the toolbox:

At Mark 22: *lrsave* and *lisave* were removed from the interface.

#### 3 Description

`nag_pde_1d_parab_remesh_fd` (d03pp) integrates the system of parabolic-elliptic equations and coupled ODEs

$$\sum_{j=1}^{\text{npde}} P_{i,j} \frac{\partial U_j}{\partial t} + Q_i = x^{-m} \frac{\partial}{\partial x} (x^m R_i), \quad i = 1, 2, \dots, \text{npde}, \quad a \leq x \leq b, t \geq t_0, \quad (1)$$

$$F_i(t, V, \dot{V}, \xi, U^*, U_x^*, R^*, U_t^*, U_{xt}^*) = 0, \quad i = 1, 2, \dots, \text{ncode}, \quad (2)$$

where (1) defines the PDE part and (2) generalizes the coupled ODE part of the problem.

In (1),  $P_{i,j}$  and  $R_i$  depend on  $x, t, U, U_x$ , and  $V$ ;  $Q_i$  depends on  $x, t, U, U_x, V$  and **linearly** on  $\dot{V}$ . The vector  $U$  is the set of PDE solution values

$$U(x, t) = [U_1(x, t), \dots, U_{\text{npde}}(x, t)]^T,$$

and the vector  $U_x$  is the partial derivative with respect to  $x$ . The vector  $V$  is the set of ODE solution values

$$V(t) = [V_1(t), \dots, V_{\text{ncode}}(t)]^T,$$

and  $\dot{V}$  denotes its derivative with respect to time.

In (2),  $\xi$  represents a vector of  $n_\xi$  spatial coupling points at which the ODEs are coupled to the PDEs. These points may or may not be equal to some of the PDE spatial mesh points.  $U^*, U_x^*, R^*, U_t^*$  and  $U_{xt}^*$  are the functions  $U, U_x, R, U_t$  and  $U_{xt}$  evaluated at these coupling points. Each  $F_i$  may only depend

linearly on time derivatives. Hence the equation (2) may be written more precisely as

$$F = G - AV - B \begin{pmatrix} U_t^* \\ U_{xt}^* \end{pmatrix}, \quad (3)$$

where  $F = [F_1, \dots, F_{\mathbf{ncode}}]^T$ ,  $G$  is a vector of length  $\mathbf{ncode}$ ,  $A$  is an  $\mathbf{ncode}$  by  $\mathbf{ncode}$  matrix,  $B$  is an  $\mathbf{ncode}$  by  $(n_\xi \times \mathbf{npde})$  matrix and the entries in  $G$ ,  $A$  and  $B$  may depend on  $t$ ,  $\xi$ ,  $U^*$ ,  $U_x^*$  and  $V$ . In practice you only need to supply a vector of information to define the ODEs and not the matrices  $A$  and  $B$ . (See Section 5 for the specification of **odedef**.)

The integration in time is from  $t_0$  to  $t_{\text{out}}$ , over the space interval  $a \leq x \leq b$ , where  $a = x_1$  and  $b = x_{\mathbf{npts}}$  are the leftmost and rightmost points of a mesh  $x_1, x_2, \dots, x_{\mathbf{npts}}$  defined initially by you and (possibly) adapted automatically during the integration according to user-specified criteria. The coordinate system in space is defined by the following values of  $m$ ;  $m = 0$  for Cartesian coordinates,  $m = 1$  for cylindrical polar coordinates and  $m = 2$  for spherical polar coordinates.

The PDE system which is defined by the functions  $P_{i,j}$ ,  $Q_i$  and  $R_i$  must be specified in **pdedef**.

The initial ( $t = t_0$ ) values of the functions  $U(x, t)$  and  $V(t)$  must be specified in **uinit**. Note that **uinit** will be called again following any initial remeshing, and so  $U(x, t_0)$  should be specified for **all** values of  $x$  in the interval  $a \leq x \leq b$ , and not just the initial mesh points.

The functions  $R_i$  which may be thought of as fluxes, are also used in the definition of the boundary conditions. The boundary conditions must have the form

$$\beta_i(x, t)R_i(x, t, U, U_x, V) = \gamma_i(x, t, U, U_x, V, \dot{V}), \quad i = 1, 2, \dots, \mathbf{npde}, \quad (4)$$

where  $x = a$  or  $x = b$ .

The boundary conditions must be specified in **bdary**. The function  $\gamma_i$  may depend **linearly** on  $\dot{V}$ .

The problem is subject to the following restrictions:

- (i) In (1),  $\dot{V}_j(t)$ , for  $j = 1, 2, \dots, \mathbf{ncode}$ , may only appear **linearly** in the functions  $Q_i$ , for  $i = 1, 2, \dots, \mathbf{npde}$ , with a similar restriction for  $\gamma$ ;
- (ii)  $P_{i,j}$  and the flux  $R_i$  must not depend on any time derivatives;
- (iii)  $t_0 < t_{\text{out}}$ , so that integration is in the forward direction;
- (iv) The evaluation of the terms  $P_{i,j}$ ,  $Q_i$  and  $R_i$  is done approximately at the mid-points of the mesh  $\mathbf{x}(i)$ , for  $i = 1, 2, \dots, \mathbf{npts}$ , by calling the **pdedef** for each mid-point in turn. Any discontinuities in these functions **must** therefore be at one or more of the fixed mesh points specified by **xfix**;
- (v) At least one of the functions  $P_{i,j}$  must be nonzero so that there is a time derivative present in the PDE problem;
- (vi) If  $m > 0$  and  $x_1 = 0.0$ , which is the left boundary point, then it must be ensured that the PDE solution is bounded at this point. This can be done by either specifying the solution at  $x = 0.0$  or by specifying a zero flux there, that is  $\beta_i = 1.0$  and  $\gamma_i = 0.0$ . See also Section 9.

The algebraic-differential equation system which is defined by the functions  $F_i$  must be specified in **odedef**. You must also specify the coupling points  $\xi$  in the array **xi**.

The parabolic equations are approximated by a system of ODEs in time for the values of  $U_i$  at mesh points. For simple problems in Cartesian coordinates, this system is obtained by replacing the space derivatives by the usual central, three-point finite difference formula. However, for polar and spherical problems, or problems with nonlinear coefficients, the space derivatives are replaced by a modified three-point formula which maintains second order accuracy. In total there are  $\mathbf{npde} \times \mathbf{npts} + \mathbf{ncode}$  ODEs in time direction. This system is then integrated forwards in time using a Backward Differentiation Formula (BDF) or a Theta method.

The adaptive space remeshing can be used to generate meshes that automatically follow the changing time-dependent nature of the solution, generally resulting in a more efficient and accurate solution using fewer mesh points than may be necessary with a fixed uniform or non-uniform mesh. Problems with travelling wavefronts or variable-width boundary layers for example will benefit from using a moving

adaptive mesh. The discrete time-step method used here (developed by Furzeland (1984)) automatically creates a new mesh based on the current solution profile at certain time-steps, and the solution is then interpolated onto the new mesh and the integration continues.

The method requires you to supply a **monitf** which specifies in an analytical or numerical form the particular aspect of the solution behaviour you wish to track. This so-called monitor function is used to choose a mesh which equally distributes the integral of the monitor function over the domain. A typical choice of monitor function is the second space derivative of the solution value at each point (or some combination of the second space derivatives if there is more than one solution component), which results in refinement in regions where the solution gradient is changing most rapidly.

You must specify the frequency of mesh updates together with certain other criteria such as adjacent mesh ratios. Remeshing can be expensive and you are encouraged to experiment with the different options in order to achieve an efficient solution which adequately tracks the desired features of the solution.

Note that unless the monitor function for the initial solution values is zero at all user-specified initial mesh points, a new initial mesh is calculated and adopted according to the user-specified remeshing criteria. **uvinit** will then be called again to determine the initial solution values at the new mesh points (there is no interpolation at this stage) and the integration proceeds.

## 4 References

Berzins M (1990) Developments in the NAG Library software for parabolic equations *Scientific Software Systems* (eds J C Mason and M G Cox) 59–72 Chapman and Hall

Berzins M, Dew P M and Furzeland R M (1989) Developing software for time-dependent problems using the method of lines and differential-algebraic integrators *Appl. Numer. Math.* **5** 375–397

Berzins M and Furzeland R M (1992) An adaptive theta method for the solution of stiff and nonstiff differential equations *Appl. Numer. Math.* **9** 1–19

Furzeland R M (1984) The construction of adaptive space meshes *TNER.85.022* Thornton Research Centre, Chester

Skeel R D and Berzins M (1990) A method for the spatial discretization of parabolic equations in one space variable *SIAM J. Sci. Statist. Comput.* **11(1)** 1–32

## 5 Parameters

### 5.1 Compulsory Input Parameters

1: **npde** – INTEGER

The number of PDEs to be solved.

*Constraint:* **npde**  $\geq$  1.

2: **m** – INTEGER

The coordinate system used:

**m** = 0

Indicates Cartesian coordinates.

**m** = 1

Indicates cylindrical polar coordinates.

**m** = 2

Indicates spherical polar coordinates.

*Constraint:* **m** = 0, 1 or 2.

3: **ts** – REAL (KIND=nag\_wp)

The initial value of the independent variable  $t$ .

*Constraint:* **ts** < **tout**.

4: **tout** – REAL (KIND=nag\_wp)

The final value of  $t$  to which the integration is to be carried out.

5: **pdedef** – SUBROUTINE, supplied by the user.

**pdedef** must evaluate the functions  $P_{i,j}$ ,  $Q_i$  and  $R_i$  which define the system of PDEs. The functions may depend on  $x$ ,  $t$ ,  $U$ ,  $U_x$  and  $V$ .  $Q_i$  may depend linearly on  $\dot{V}$ . **pdedef** is called approximately midway between each pair of mesh points in turn by nag\_pde\_1d\_parab\_remesh\_fd (d03pp).

```
[p, q, r, ires, user] = pdedef(npde, t, x, u, ux, ncode, v, vdot, ires, user)
```

### Input Parameters

1: **npde** – INTEGER

The number of PDEs in the system.

2: **t** – REAL (KIND=nag\_wp)

The current value of the independent variable  $t$ .

3: **x** – REAL (KIND=nag\_wp)

The current value of the space variable  $x$ .

4: **u(npde)** – REAL (KIND=nag\_wp) array

**u**( $i$ ) contains the value of the component  $U_i(x, t)$ , for  $i = 1, 2, \dots, \mathbf{npde}$ .

5: **ux(npde)** – REAL (KIND=nag\_wp) array

**ux**( $i$ ) contains the value of the component  $\frac{\partial U_i(x, t)}{\partial x}$ , for  $i = 1, 2, \dots, \mathbf{npde}$ .

6: **ncode** – INTEGER

The number of coupled ODEs in the system.

7: **v(ncode)** – REAL (KIND=nag\_wp) array

If **ncode** > 0, **v**( $i$ ) contains the value of the component  $V_i(t)$ , for  $i = 1, 2, \dots, \mathbf{ncode}$ .

8: **vdot(ncode)** – REAL (KIND=nag\_wp) array

If **ncode** > 0, **vdot**( $i$ ) contains the value of component  $\dot{V}_i(t)$ , for  $i = 1, 2, \dots, \mathbf{ncode}$ .

**Note:**  $\dot{V}_i(t)$ , for  $i = 1, 2, \dots, \mathbf{ncode}$ , may only appear linearly in  $Q_j$ , for  $j = 1, 2, \dots, \mathbf{npde}$ .

9: **ires** – INTEGER

Set to  $-1$  or  $1$ .

10: **user** – INTEGER array

**pdedef** is called from `nag_pde_1d_parab_remesh_fd` (d03pp) with the object supplied to `nag_pde_1d_parab_remesh_fd` (d03pp).

#### Output Parameters

1: **p(npde, npde)** – REAL (KIND=nag\_wp) array

**p**( $i, j$ ) must be set to the value of  $P_{i,j}(x, t, U, U_x, V)$ , for  $i = 1, 2, \dots, \mathbf{npde}$  and  $j = 1, 2, \dots, \mathbf{npde}$ .

2: **q(npde)** – REAL (KIND=nag\_wp) array

**q**( $i$ ) must be set to the value of  $Q_i(x, t, U, U_x, V, \dot{V})$ , for  $i = 1, 2, \dots, \mathbf{npde}$ .

3: **r(npde)** – REAL (KIND=nag\_wp) array

**r**( $i$ ) must be set to the value of  $R_i(x, t, U, U_x, V)$ , for  $i = 1, 2, \dots, \mathbf{npde}$ .

4: **ires** – INTEGER

Should usually remain unchanged. However, you may set **ires** to force the integration function to take certain actions as described below:

**ires** = 2

Indicates to the integrator that control should be passed back immediately to the calling (sub)routine with the error indicator set to **ifail** = 6.

**ires** = 3

Indicates to the integrator that the current time step should be abandoned and a smaller time step used instead. You may wish to set **ires** = 3 when a physically meaningless input or output value has been generated. If you consecutively set **ires** = 3, then `nag_pde_1d_parab_remesh_fd` (d03pp) returns to the calling function with the error indicator set to **ifail** = 4.

5: **user** – INTEGER array

6: **boundary** – SUBROUTINE, supplied by the user.

**boundary** must evaluate the functions  $\beta_i$  and  $\gamma_i$  which describe the boundary conditions, as given in (4).

```
[beta, gamma, ires, user] = boundary(npde, t, u, ux, ncode, v, vdot, ibnd,
ires, user)
```

#### Input Parameters

1: **npde** – INTEGER

The number of PDEs in the system.

2: **t** – REAL (KIND=nag\_wp)

The current value of the independent variable  $t$ .

3: **u(npde)** – REAL (KIND=nag\_wp) array

**u**( $i$ ) contains the value of the component  $U_i(x, t)$  at the boundary specified by **ibnd**, for  $i = 1, 2, \dots, \mathbf{npde}$ .

- 4: **ux(npde)** – REAL (KIND=nag\_wp) array  
**ux**(*i*) contains the value of the component  $\frac{\partial U_i(x,t)}{\partial x}$  at the boundary specified by **ibnd**, for  $i = 1, 2, \dots, \mathbf{npde}$ .
- 5: **ncode** – INTEGER  
The number of coupled ODEs in the system.
- 6: **v(ncode)** – REAL (KIND=nag\_wp) array  
If **ncode** > 0, **v**(*i*) contains the value of the component  $V_i(t)$ , for  $i = 1, 2, \dots, \mathbf{ncode}$ .
- 7: **vdot(ncode)** – REAL (KIND=nag\_wp) array  
**vdot**(*i*) contains the value of component  $\dot{V}_i(t)$ , for  $i = 1, 2, \dots, \mathbf{ncode}$ .  
**Note:**  $\dot{V}_i(t)$ , for  $i = 1, 2, \dots, \mathbf{ncode}$ , may only appear linearly in  $\gamma_j$ , for  $j = 1, 2, \dots, \mathbf{npde}$ .
- 8: **ibnd** – INTEGER  
Specifies which boundary conditions are to be evaluated.  
**ibnd** = 0  
**bdary** must set up the coefficients of the left-hand boundary,  $x = a$ .  
**ibnd**  $\neq$  0  
**bdary** must set up the coefficients of the right-hand boundary,  $x = b$ .
- 9: **ires** – INTEGER  
Set to -1 or 1.
- 10: **user** – INTEGER array  
**bdary** is called from nag\_pde\_1d\_parab\_remesh\_fd (d03pp) with the object supplied to nag\_pde\_1d\_parab\_remesh\_fd (d03pp).

### Output Parameters

- 1: **beta(npde)** – REAL (KIND=nag\_wp) array  
**beta**(*i*) must be set to the value of  $\beta_i(x,t)$  at the boundary specified by **ibnd**, for  $i = 1, 2, \dots, \mathbf{npde}$ .
- 2: **gamma(npde)** – REAL (KIND=nag\_wp) array  
**gamma**(*i*) must be set to the value of  $\gamma_i(x,t,U,U_x,V,\dot{V})$  at the boundary specified by **ibnd**, for  $i = 1, 2, \dots, \mathbf{npde}$ .
- 3: **ires** – INTEGER  
Should usually remain unchanged. However, you may set **ires** to force the integration function to take certain actions as described below:  
**ires** = 2  
Indicates to the integrator that control should be passed back immediately to the calling (sub)routine with the error indicator set to **ifail** = 6.  
**ires** = 3  
Indicates to the integrator that the current time step should be abandoned and a smaller time step used instead. You may wish to set **ires** = 3 when a physically meaningless input or output value has been generated. If you consecutively set

**ires** = 3, then `nag_pde_1d_parab_remesh_fd` (d03pp) returns to the calling function with the error indicator set to **ifail** = 4.

4: **user** – INTEGER array

7: **uvinit** – SUBROUTINE, supplied by the user.

**uvinit** must supply the initial ( $t = t_0$ ) values of  $U(x, t)$  and  $V(t)$  for all values of  $x$  in the interval  $a \leq x \leq b$ .

```
[u, v, user] = uvinit(npde, npts, nxi, x, xi, ncode, user)
```

#### Input Parameters

1: **npde** – INTEGER

The number of PDEs in the system.

2: **npts** – INTEGER

The number of mesh points in the interval  $[a, b]$ .

3: **nxi** – INTEGER

The number of ODE/PDE coupling points.

4: **x(npts)** – REAL (KIND=nag\_wp) array

The current mesh.  $\mathbf{x}(i)$  contains the value of  $x_i$ , for  $i = 1, 2, \dots, \mathbf{npts}$ .

5: **xi(nxi)** – REAL (KIND=nag\_wp) array

If  $\mathbf{nxi} > 0$ ,  $\mathbf{xi}(i)$  contains the value of the ODE/PDE coupling point,  $\xi_i$ , for  $i = 1, 2, \dots, \mathbf{nxi}$ .

6: **ncode** – INTEGER

The number of coupled ODEs in the system.

7: **user** – INTEGER array

**uvinit** is called from `nag_pde_1d_parab_remesh_fd` (d03pp) with the object supplied to `nag_pde_1d_parab_remesh_fd` (d03pp).

#### Output Parameters

1: **u(npde, npts)** – REAL (KIND=nag\_wp) array

If  $\mathbf{nxi} > 0$ ,  $\mathbf{u}(i, j)$  contains the value of the component  $U_i(x_j, t_0)$ , for  $i = 1, 2, \dots, \mathbf{npde}$  and  $j = 1, 2, \dots, \mathbf{npts}$ .

2: **v(ncode)** – REAL (KIND=nag\_wp) array

$\mathbf{v}(i)$  contains the value of component  $V_i(t_0)$ , for  $i = 1, 2, \dots, \mathbf{ncode}$ .

3: **user** – INTEGER array

8: **u(neqn)** – REAL (KIND=nag\_wp) array

If **ind** = 1, the value of **u** must be unchanged from the previous call.

9: **x(npts)** – REAL (KIND=nag\_wp) array

The initial mesh points in the space direction. **x**(1) must specify the left-hand boundary,  $a$ , and **x**(npts) must specify the right-hand boundary,  $b$ .

*Constraint:* **x**(1) < **x**(2) < ... < **x**(npts).

10: **ncode** – INTEGER

The number of coupled ODE in the system.

*Constraint:* **ncode**  $\geq$  0.

11: **odedef** – SUBROUTINE, supplied by the NAG Library or the user.

**odedef** must evaluate the functions  $F$ , which define the system of ODEs, as given in (3).

If you wish to compute the solution of a system of PDEs only (**ncode** = 0), **odedef** must be the string nag\_pde\_1d\_parab\_remesh\_fd\_dummy\_odedef (d53pck).

```
[f, ires, user] = odedef(npde, t, ncode, v, vdot, nxi, xi, ucp, ucpx, rcp,
ucpt, ucptx, ires, user)
```

#### Input Parameters

1: **npde** – INTEGER

The number of PDEs in the system.

2: **t** – REAL (KIND=nag\_wp)

The current value of the independent variable  $t$ .

3: **ncode** – INTEGER

The number of coupled ODEs in the system.

4: **v(ncode)** – REAL (KIND=nag\_wp) array

If **ncode** > 0, **v**( $i$ ) contains the value of the component  $V_i(t)$ , for  $i = 1, 2, \dots, \mathbf{ncode}$ .

5: **vdot(ncode)** – REAL (KIND=nag\_wp) array

If **ncode** > 0, **vdot**( $i$ ) contains the value of component  $\dot{V}_i(t)$ , for  $i = 1, 2, \dots, \mathbf{ncode}$ .

6: **nxi** – INTEGER

The number of ODE/PDE coupling points.

7: **xi(nxi)** – REAL (KIND=nag\_wp) array

If **nxi** > 0, **xi**( $i$ ) contains the ODE/PDE coupling points,  $\xi_i$ , for  $i = 1, 2, \dots, \mathbf{nxi}$ .

8: **ucp(npde,:)** – REAL (KIND=nag\_wp) array

The second dimension of the array **ucp** must be at least  $\max(1, \mathbf{nxi})$ .

If **nxi** > 0, **ucp**( $i, j$ ) contains the value of  $U_i(x, t)$  at the coupling point  $x = \xi_j$ , for  $i = 1, 2, \dots, \mathbf{npde}$  and  $j = 1, 2, \dots, \mathbf{nxi}$ .

9: **ucpx(npde,:)** – REAL (KIND=nag\_wp) array

The second dimension of the array **ucpx** must be at least  $\max(1, \mathbf{nxi})$ .

If  $\mathbf{nxi} > 0$ ,  $\mathbf{ucpx}(i, j)$  contains the value of  $\frac{\partial U_i(x, t)}{\partial x}$  at the coupling point  $x = \xi_j$ , for  $i = 1, 2, \dots, \mathbf{npde}$  and  $j = 1, 2, \dots, \mathbf{nxi}$ .

10: **rcp(npde, :)** – REAL (KIND=nag\_wp) array

The second dimension of the array **rcp** must be at least  $\max(1, \mathbf{nxi})$ .

**rcp**( $i, j$ ) contains the value of the flux  $R_i$  at the coupling point  $x = \xi_j$ , for  $i = 1, 2, \dots, \mathbf{npde}$  and  $j = 1, 2, \dots, \mathbf{nxi}$ .

11: **ucpt(npde, :)** – REAL (KIND=nag\_wp) array

The second dimension of the array **ucpt** must be at least  $\max(1, \mathbf{nxi})$ .

If  $\mathbf{nxi} > 0$ , **ucpt**( $i, j$ ) contains the value of  $\frac{\partial U_i}{\partial t}$  at the coupling point  $x = \xi_j$ , for  $i = 1, 2, \dots, \mathbf{npde}$  and  $j = 1, 2, \dots, \mathbf{nxi}$ .

12: **ucptx(npde, :)** – REAL (KIND=nag\_wp) array

The second dimension of the array **ucptx** must be at least  $\max(1, \mathbf{nxi})$ .

**ucptx**( $i, j$ ) contains the value of  $\frac{\partial^2 U_i}{\partial x \partial t}$  at the coupling point  $x = \xi_j$ , for  $i = 1, 2, \dots, \mathbf{npde}$  and  $j = 1, 2, \dots, \mathbf{nxi}$ .

13: **ires** – INTEGER

The form of  $F$  that must be returned in the array **f**.

**ires** = 1

Equation (5) must be used.

**ires** = -1

Equation (6) must be used.

14: **user** – INTEGER array

**odedef** is called from `nag_pde_1d_parab_remesh_fd` (d03pp) with the object supplied to `nag_pde_1d_parab_remesh_fd` (d03pp).

### Output Parameters

1: **f(ncode)** – REAL (KIND=nag\_wp) array

**f**( $i$ ) must contain the  $i$ th component of  $F$ , for  $i = 1, 2, \dots, \mathbf{ncode}$ , where  $F$  is defined as

$$F = G - A\dot{V} - B \begin{pmatrix} U_t^* \\ U_{xt}^* \end{pmatrix}, \quad (5)$$

or

$$F = -A\dot{V} - B \begin{pmatrix} U_t^* \\ U_{xt}^* \end{pmatrix}. \quad (6)$$

The definition of  $F$  is determined by the input value of **ires**.

2: **ires** – INTEGER

Should usually remain unchanged. However, you may reset **ires** to force the integration function to take certain actions as described below:

**ires** = 2

Indicates to the integrator that control should be passed back immediately to the calling (sub)routines with the error indicator set to **ifail** = 6.

**ires** = 3

Indicates to the integrator that the current time step should be abandoned and a smaller time step used instead. You may wish to set **ires** = 3 when a physically meaningless input or output value has been generated. If you consecutively set **ires** = 3, then `nag_pde_1d_parab_remesh_fd` (d03pp) returns to the calling function with the error indicator set to **ifail** = 4.

3: **user** – INTEGER array

12: **xi(nxi)** – REAL (KIND=nag\_wp) array

If **nxi** > 0, **xi**(*i*), for *i* = 1, 2, ..., **nxi**, must be set to the ODE/PDE coupling points.

*Constraint:*  $\mathbf{x}(1) \leq \mathbf{xi}(1) < \mathbf{xi}(2) < \dots < \mathbf{xi}(\mathbf{nxi}) \leq \mathbf{x}(\mathbf{npts})$ .

13: **rtol**(:) – REAL (KIND=nag\_wp) array

The dimension of the array **rtol** must be at least 1 if **itol** = 1 or 2 and at least **neqn** if **itol** = 3 or 4

The relative local error tolerance.

*Constraint:*  $\mathbf{rtol}(i) \geq 0.0$  for all relevant *i*.

14: **atol**(:) – REAL (KIND=nag\_wp) array

The dimension of the array **atol** must be at least 1 if **itol** = 1 or 3 and at least **neqn** if **itol** = 2 or 4

The absolute local error tolerance.

*Constraints:*

$\mathbf{atol}(i) \geq 0.0$  for all relevant *i*;

Corresponding elements of **atol** and **rtol** cannot both be 0.0.

15: **itol** – INTEGER

A value to indicate the form of the local error test. **itol** indicates to `nag_pde_1d_parab_remesh_fd` (d03pp) whether to interpret either or both of **rtol** or **atol** as a vector or scalar. The error test to be satisfied is  $\|e_i/w_i\| < 1.0$ , where  $w_i$  is defined as follows:

<b>itol</b>	<b>rtol</b>	<b>atol</b>	$w_i$
1	scalar	scalar	$\mathbf{rtol}(1) \times  U_i  + \mathbf{atol}(1)$
2	scalar	vector	$\mathbf{rtol}(1) \times  U_i  + \mathbf{atol}(i)$
3	vector	scalar	$\mathbf{rtol}(i) \times  U_i  + \mathbf{atol}(1)$
4	vector	vector	$\mathbf{rtol}(i) \times  U_i  + \mathbf{atol}(i)$

In the above,  $e_i$  denotes the estimated local error for the *i*th component of the coupled PDE/ODE system in time, **u**(*i*), for *i* = 1, 2, ..., **neqn**.

The choice of norm used is defined by the argument **norm\_p**.

*Constraint:*  $1 \leq \mathbf{itol} \leq 4$ .

16: **norm\_p** – CHARACTER(1)

The type of norm to be used.

**norm\_p** = 'M'

Maximum norm.

**norm\_p** = 'A'

Averaged  $L_2$  norm.

If  $\mathbf{u}_{\text{norm}}$  denotes the norm of the vector  $\mathbf{u}$  of length  $\mathbf{neqn}$ , then for the averaged  $L_2$  norm

$$\mathbf{u}_{\text{norm}} = \sqrt{\frac{1}{\mathbf{neqn}} \sum_{i=1}^{\mathbf{neqn}} (\mathbf{u}(i)/w_i)^2},$$

while for the maximum norm

$$\mathbf{u}_{\text{norm}} = \max_i |\mathbf{u}(i)/w_i|.$$

See the description of **itol** for the formulation of the weight vector  $w$ .

*Constraint:* **norm\_p** = 'M' or 'A'.

17: **laopt** – CHARACTER(1)

The type of matrix algebra required.

**laopt** = 'F'

Full matrix methods to be used.

**laopt** = 'B'

Banded matrix methods to be used.

**laopt** = 'S'

Sparse matrix methods to be used.

*Constraint:* **laopt** = 'F', 'B' or 'S'.

**Note:** you are recommended to use the banded option when no coupled ODEs are present (i.e., **ncode** = 0).

18: **algopt(30)** – REAL (KIND=nag\_wp) array

May be set to control various options available in the integrator. If you wish to employ all the default options, then **algopt(1)** should be set to 0.0. Default values will also be used for any other elements of **algopt** set to zero. The permissible values, default values, and meanings are as follows:

**algopt(1)**

Selects the ODE integration method to be used. If **algopt(1)** = 1.0, a BDF method is used and if **algopt(1)** = 2.0, a Theta method is used. The default value is **algopt(1)** = 1.0.

If **algopt(1)** = 2.0, then **algopt(i)**, for  $i = 2, 3, 4$  are not used.

**algopt(2)**

Specifies the maximum order of the BDF integration formula to be used. **algopt(2)** may be 1.0, 2.0, 3.0, 4.0 or 5.0. The default value is **algopt(2)** = 5.0.

**algopt(3)**

Specifies what method is to be used to solve the system of nonlinear equations arising on each step of the BDF method. If **algopt(3)** = 1.0 a modified Newton iteration is used and if **algopt(3)** = 2.0 a functional iteration method is used. If functional iteration is selected and the integrator encounters difficulty, then there is an automatic switch to the modified Newton iteration. The default value is **algopt(3)** = 1.0.

**algopt(4)**

Specifies whether or not the Petzold error test is to be employed. The Petzold error test results in extra overhead but is more suitable when algebraic equations are present, such as  $P_{i,j} = 0.0$ , for  $j = 1, 2, \dots, \mathbf{npde}$ , for some  $i$  or when there is no  $\dot{V}_i(t)$  dependence in the coupled ODE system. If **algopt(4)** = 1.0, then the Petzold test is used. If **algopt(4)** = 2.0, then the Petzold test is not used. The default value is **algopt(4)** = 1.0.

If **algot**(1) = 1.0, then **algot**( $i$ ), for  $i = 5, 6, 7$ , are not used.

**algot**(5)

Specifies the value of Theta to be used in the Theta integration method.  $0.51 \leq \mathbf{algot}(5) \leq 0.99$ . The default value is **algot**(5) = 0.55.

**algot**(6)

Specifies what method is to be used to solve the system of nonlinear equations arising on each step of the Theta method. If **algot**(6) = 1.0, a modified Newton iteration is used and if **algot**(6) = 2.0, a functional iteration method is used. The default value is **algot**(6) = 1.0.

**algot**(7)

Specifies whether or not the integrator is allowed to switch automatically between modified Newton and functional iteration methods in order to be more efficient. If **algot**(7) = 1.0, then switching is allowed and if **algot**(7) = 2.0, then switching is not allowed. The default value is **algot**(7) = 1.0.

**algot**(11)

Specifies a point in the time direction,  $t_{\text{crit}}$ , beyond which integration must not be attempted. The use of  $t_{\text{crit}}$  is described under the argument **itask**. If **algot**(1)  $\neq$  0.0, a value of 0.0 for **algot**(11), say, should be specified even if **itask** subsequently specifies that  $t_{\text{crit}}$  will not be used.

**algot**(12)

Specifies the minimum absolute step size to be allowed in the time integration. If this option is not required, **algot**(12) should be set to 0.0.

**algot**(13)

Specifies the maximum absolute step size to be allowed in the time integration. If this option is not required, **algot**(13) should be set to 0.0.

**algot**(14)

Specifies the initial step size to be attempted by the integrator. If **algot**(14) = 0.0, then the initial step size is calculated internally.

**algot**(15)

Specifies the maximum number of steps to be attempted by the integrator in any one call. If **algot**(15) = 0.0, then no limit is imposed.

**algot**(23)

Specifies what method is to be used to solve the nonlinear equations at the initial point to initialize the values of  $U$ ,  $U_t$ ,  $V$  and  $\dot{V}$ . If **algot**(23) = 1.0, a modified Newton iteration is used and if **algot**(23) = 2.0, functional iteration is used. The default value is **algot**(23) = 1.0.

**algot**(29) and **algot**(30) are used only for the sparse matrix algebra option, **laopt** = 'S'.

**algot**(29)

Governs the choice of pivots during the decomposition of the first Jacobian matrix. It should lie in the range  $0.0 < \mathbf{algot}(29) < 1.0$ , with smaller values biasing the algorithm towards maintaining sparsity at the expense of numerical stability. If **algot**(29) lies outside this range then the default value is used. If the functions regard the Jacobian matrix as numerically singular then increasing **algot**(29) towards 1.0 may help, but at the cost of increased fill-in. The default value is **algot**(29) = 0.1.

**algot**(30)

Is used as a relative pivot threshold during subsequent Jacobian decompositions (see **algot**(29)) below which an internal error is invoked. If **algot**(30) is greater than 1.0 no check is made on the pivot size, and this may be a necessary option if the Jacobian is found to be numerically singular (see **algot**(29)). The default value is **algot**(30) = 0.0001.

19: **remesh** – LOGICAL

Indicates whether or not spatial remeshing should be performed.

**remesh** = *true*

Indicates that spatial remeshing should be performed as specified.

**remesh** = *false*

Indicates that spatial remeshing should be suppressed.

**Note:** **remesh** should **not** be changed between consecutive calls to `nag_pde_1d_parab_remesh_fd` (d03pp). Remeshing can be switched off or on at specified times by using appropriate values for the arguments **nrmesh** and **trmesh** at each call.

20: **xfix**(:) – REAL (KIND=nag\_wp) array

The dimension of the array **xfix** must be at least  $\max(1, \mathbf{nxfix})$

**xfix**(*i*), for  $i = 1, 2, \dots, \mathbf{nxfix}$ , must contain the value of the *x* coordinate at the *i*th fixed mesh point.

*Constraints:*

$\mathbf{xfix}(i) < \mathbf{xfix}(i + 1)$ , for  $i = 1, 2, \dots, \mathbf{nxfix} - 1$ ;

each fixed mesh point must coincide with a user-supplied initial mesh point, that is  $\mathbf{xfix}(i) = \mathbf{x}(j)$  for some  $j$ ,  $2 \leq j \leq \mathbf{npts} - 1$ .

**Note:** the positions of the fixed mesh points in the array **x** remain fixed during remeshing, and so the number of mesh points between adjacent fixed points (or between fixed points and end points) does not change. You should take this into account when choosing the initial mesh distribution.

21: **nrmesh** – INTEGER

Specifies the spatial remeshing frequency and criteria for the calculation and adoption of a new mesh.

**nrmesh** < 0

Indicates that a new mesh is adopted according to the argument **dxmesh**. The mesh is tested every  $|\mathbf{nrmesh}|$  timesteps.

**nrmesh** = 0

Indicates that remeshing should take place just once at the end of the first time step reached when  $t > \mathbf{trmesh}$ .

**nrmesh** > 0

Indicates that remeshing will take place every **nrmesh** time steps, with no testing using **dxmesh**.

**Note:** **nrmesh** may be changed between consecutive calls to `nag_pde_1d_parab_remesh_fd` (d03pp) to give greater flexibility over the times of remeshing.

22: **dxmesh** – REAL (KIND=nag\_wp)

Determines whether a new mesh is adopted when **nrmesh** is set less than zero. A possible new mesh is calculated at the end of every  $|\mathbf{nrmesh}|$  time steps, but is adopted only if

$$x_i^{(\text{new})} > x_i^{(\text{old})} + \mathbf{dxmesh} \times (x_{i+1}^{(\text{old})} - x_i^{(\text{old})})$$

or

$$x_i^{(\text{new})} < x_i^{(\text{old})} - \mathbf{dxmesh} \times (x_i^{(\text{old})} - x_{i-1}^{(\text{old})})$$

**dxmesh** thus imposes a lower limit on the difference between one mesh and the next.

*Constraint:* **dxmesh**  $\geq 0.0$ .

23: **trmesh** – REAL (KIND=nag\_wp)

Specifies when remeshing will take place when **nrmesh** is set to zero. Remeshing will occur just once at the end of the first time step reached when  $t$  is greater than **trmesh**.

**Note:** **trmesh** may be changed between consecutive calls to `nag_pde_1d_parab_remesh_fd` (d03pp) to force remeshing at several specified times.

24: **ipminf** – INTEGER

The level of trace information regarding the adaptive remeshing. Details are directed to the current advisory message unit (see `nag_file_set_unit_advisory` (x04ab)).

**ipminf** = 0

No trace information.

**ipminf** = 1

Brief summary of mesh characteristics.

**ipminf** = 2

More detailed information, including old and new mesh points, mesh sizes and monitor function values.

*Constraint:* **ipminf** = 0, 1 or 2.

25: **monitf** – SUBROUTINE, supplied by the NAG Library or the user.

**monitf** must supply and evaluate a remesh monitor function to indicate the solution behaviour of interest.

If you specify **remesh** = *false*, i.e., no remeshing, then **monitf** will not be called and the string `nag_pde_1d_parab_remesh_fd_dummy_monitf` (d53pcl) may be used for **monitf**.

```
[fmon, user] = monitf(t, npts, npde, x, u, r, user)
```

#### Input Parameters

1: **t** – REAL (KIND=nag\_wp)

The current value of the independent variable  $t$ .

2: **npts** – INTEGER

The number of mesh points in the interval  $[a, b]$ .

3: **npde** – INTEGER

The number of PDEs in the system.

4: **x(npts)** – REAL (KIND=nag\_wp) array

The current mesh.  $\mathbf{x}(i)$  contains the value of  $x_i$ , for  $i = 1, 2, \dots, \mathbf{npts}$ .

5: **u(npde, npts)** – REAL (KIND=nag\_wp) array

$\mathbf{u}(i, j)$  contains the value of  $U_i(x, t)$  at  $x = \mathbf{x}(j)$  and time  $t$ , for  $i = 1, 2, \dots, \mathbf{npde}$  and  $j = 1, 2, \dots, \mathbf{npts}$ .

6: **r(npde, npts)** – REAL (KIND=nag\_wp) array

$\mathbf{r}(i, j)$  contains the value of  $R_i(x, t, U, U_x, V)$  at  $x = \mathbf{x}(j)$  and time  $t$ , for  $i = 1, 2, \dots, \mathbf{npde}$  and  $j = 1, 2, \dots, \mathbf{npts}$ .

7: **user** – INTEGER array  
**monitf** is called from `nag_pde_1d_parab_remesh_fd` (d03pp) with the object supplied to `nag_pde_1d_parab_remesh_fd` (d03pp).

#### Output Parameters

1: **fmon(npts)** – REAL (KIND=`nag_wp`) array  
**fmon**(*i*) must contain the value of the monitor function  $F^{\text{mon}}(x)$  at mesh point  $x = \mathbf{x}(i)$ .

2: **user** – INTEGER array

26: **rsave**(*lrsave*) – REAL (KIND=`nag_wp`) array

If **ind** = 0, **rsave** need not be set on entry.

If **ind** = 1, **rsave** must be unchanged from the previous call to the function because it contains required information about the iteration.

27: **isave**(*lisave*) – INTEGER array

If **ind** = 0, **isave** need not be set on entry.

If **ind** = 1, **isave** must be unchanged from the previous call to the function because it contains required information about the iteration required for subsequent calls. In particular:

**isave**(1)

Contains the number of steps taken in time.

**isave**(2)

Contains the number of residual evaluations of the resulting ODE system used. One such evaluation involves computing the PDE functions at all the mesh points, as well as one evaluation of the functions in the boundary conditions.

**isave**(3)

Contains the number of Jacobian evaluations performed by the time integrator.

**isave**(4)

Contains the order of the ODE method last used in the time integration.

**isave**(5)

Contains the number of Newton iterations performed by the time integrator. Each iteration involves residual evaluation of the resulting ODE system followed by a back-substitution using the *LU* decomposition of the Jacobian matrix.

The rest of the array is used as workspace.

28: **itask** – INTEGER

Specifies the task to be performed by the ODE integrator.

**itask** = 1

Normal computation of output values **u** at  $t = \mathbf{tout}$ .

**itask** = 2

One step and return.

**itask** = 3

Stop at first internal integration point at or beyond  $t = \mathbf{tout}$ .

**itask** = 4

Normal computation of output values **u** at  $t = \mathbf{tout}$  but without overshooting  $t = t_{\text{crit}}$  where  $t_{\text{crit}}$  is described under the argument **algopt**.

**itask** = 5

Take one step in the time direction and return, without passing  $t_{\text{crit}}$ , where  $t_{\text{crit}}$  is described under the argument **algot**.

*Constraint:* **itask** = 1, 2, 3, 4 or 5.

29: **itrace** – INTEGER

The level of trace information required from nag\_pde\_1d\_parab\_remesh\_fd (d03pp) and the underlying ODE solver:

**itrace**  $\leq$  -1

No output is generated.

**itrace** = 0

Only warning messages from the PDE solver are printed on the current error message unit (see nag\_file\_set\_unit\_error (x04aa)).

**itrace** = 1

Output from the underlying ODE solver is printed on the current advisory message unit (see nag\_file\_set\_unit\_advisory (x04ab)). This output contains details of Jacobian entries, the nonlinear iteration and the time integration during the computation of the ODE system.

**itrace** = 2

Output from the underlying ODE solver is similar to that produced when **itrace** = 1, except that the advisory messages are given in greater detail.

**itrace**  $\geq$  3

Output from the underlying ODE solver is similar to that produced when **itrace** = 2, except that the advisory messages are given in greater detail.

You are advised to set **itrace** = 0, unless you are experienced with Sub-chapter D02M–N.

30: **ind** – INTEGER

Must be set to 0 or 1.

**ind** = 0

Starts or restarts the integration in time.

**ind** = 1

Continues the integration after an earlier exit from the function. In this case, only the arguments **tout** and **ifail** and the remeshing arguments **nrmesh**, **dxmesh**, **trmesh**, **xratio** and **con** may be reset between calls to nag\_pde\_1d\_parab\_remesh\_fd (d03pp).

*Constraint:*  $0 \leq \mathbf{ind} \leq 1$ .

31: **cwsav**(10) – CHARACTER(80) array

32: **lwsav**(100) – LOGICAL array

33: **iwsav**(505) – INTEGER array

34: **rwsav**(1100) – REAL (KIND=nag\_wp) array

## 5.2 Optional Input Parameters

1: **npts** – INTEGER

*Default:* the dimension of the array **x**.

The number of mesh points in the interval  $[a, b]$ .

*Constraint:* **npts**  $\geq$  3.

2: **nxi** – INTEGER

*Default:* the dimension of the array **xi**.

The number of ODE/PDE coupling points.

*Constraints:*

if **ncode** = 0, **nxi** = 0;  
if **ncode** > 0, **nxi** ≥ 0.

3: **neqn** – INTEGER

*Default:* the dimension of the array **u**.

The number of ODEs in the time direction.

*Constraint:* **neqn** = **npde** × **npts** + **ncode**.

4: **nxfix** – INTEGER

*Default:* the dimension of the array **xfix**.

The number of fixed mesh points.

*Constraint:*  $0 \leq \mathbf{nxfix} \leq \mathbf{npts} - 2$ .

**Note:** the end points **x**(1) and **x**(**npts**) are fixed automatically and hence should not be specified as fixed points.

5: **xratio** – REAL (KIND=nag\_wp)

*Suggested value:* **xratio** = 1.5.

*Default:* 1.5

An input bound on the adjacent mesh ratio (greater than 1.0 and typically in the range 1.5 to 3.0). The remeshing functions will attempt to ensure that

$$(x_i - x_{i-1})/\mathbf{xratio} < x_{i+1} - x_i < \mathbf{xratio} \times (x_i - x_{i-1}).$$

*Constraint:* **xratio** > 1.0.

6: **con** – REAL (KIND=nag\_wp)

*Suggested value:* **con** = 2.0/(**npts** - 1).

*Default:* 2.0/(**npts** - 1)

An input bound on the sub-integral of the monitor function  $F^{\text{mon}}(x)$  over each space step. The remeshing functions will attempt to ensure that

$$\int_{x_i}^{x_{i+1}} F^{\text{mon}}(x) dx \leq \mathbf{con} \int_{x_1}^{x_{\mathbf{npts}}} F^{\text{mon}}(x) dx,$$

(see Furzeland (1984)). **con** gives you more control over the mesh distribution e.g., decreasing **con** allows more clustering. A typical value is 2/(**npts** - 1), but you are encouraged to experiment with different values. Its value is not critical and the mesh should be qualitatively correct for all values in the range given below.

*Constraint:*  $0.1/(\mathbf{npts} - 1) \leq \mathbf{con} \leq 10.0/(\mathbf{npts} - 1)$ .

7: **user** – INTEGER array

**user** is not used by nag\_pde\_1d\_parab\_remesh\_fd (d03pp), but is passed to **pdedef**, **boundary**, **uvinit**, **odedef** and **monitf**. Note that for large objects it may be more efficient to use a global variable which is accessible from the m-files than to use **user**.

### 5.3 Output Parameters

1: **ts** – REAL (KIND=nag\_wp)

The value of  $t$  corresponding to the solution values in **u**. Normally **ts** = **tout**.

2: **u(neqn)** – REAL (KIND=nag\_wp) array

**u**( $\text{npde} \times (j - 1) + i$ ) contains the computed solution  $U_i(x_j, t)$ , for  $i = 1, 2, \dots, \text{npde}$  and  $j = 1, 2, \dots, \text{npts}$ , and **u**( $\text{npts} \times \text{npde} + k$ ) contains  $V_k(t)$ , for  $k = 1, 2, \dots, \text{ncode}$ , evaluated at  $t = \text{ts}$ .

3: **x(npts)** – REAL (KIND=nag\_wp) array

The final values of the mesh points.

4: **rsave(lrsave)** – REAL (KIND=nag\_wp) array

If **ind** = 1, **rsave** must be unchanged from the previous call to the function because it contains required information about the iteration.

5: **isave(lisave)** – INTEGER array

If **ind** = 1, **isave** must be unchanged from the previous call to the function because it contains required information about the iteration required for subsequent calls. In particular:

**isave**(1)

Contains the number of steps taken in time.

**isave**(2)

Contains the number of residual evaluations of the resulting ODE system used. One such evaluation involves computing the PDE functions at all the mesh points, as well as one evaluation of the functions in the boundary conditions.

**isave**(3)

Contains the number of Jacobian evaluations performed by the time integrator.

**isave**(4)

Contains the order of the ODE method last used in the time integration.

**isave**(5)

Contains the number of Newton iterations performed by the time integrator. Each iteration involves residual evaluation of the resulting ODE system followed by a back-substitution using the *LU* decomposition of the Jacobian matrix.

The rest of the array is used as workspace.

6: **ind** – INTEGER

**ind** = 1.

7: **user** – INTEGER array

8: **cwsav(10)** – CHARACTER(80) array

9: **lwsav(100)** – LOGICAL array

10: **iwsav(505)** – INTEGER array

11: **rwsav(1100)** – REAL (KIND=nag\_wp) array

12: **ifail** – INTEGER

**ifail** = 0 unless the function detects an error (see Section 5).

## 6 Error Indicators and Warnings

Errors or warnings detected by the function:

**ifail** = 1

On entry, **tout** – **ts** is too small,  
 or **itask**  $\neq$  1, 2, 3, 4 or 5,  
 or **m**  $\neq$  0, 1 or 2,  
 or at least one of the coupling points defined in array **xi** is outside the interval  $[\mathbf{x}(1), \mathbf{x}(\mathbf{npts})]$ ,  
 or **m**  $>$  0 and  $\mathbf{x}(1) <$  0.0,  
 or **npts**  $<$  3,  
 or **npde**  $<$  1,  
 or **norm\_p**  $\neq$  'A' or 'M',  
 or **laopt**  $\neq$  'F', 'B' or 'S',  
 or **itol**  $\neq$  1, 2, 3 or 4,  
 or **ind**  $\neq$  0 or 1,  
 or mesh points  $\mathbf{x}(i)$  are badly ordered,  
 or *lrsave* is too small,  
 or *lisave* is too small,  
 or **ncode** and **nxi** are incorrectly defined,  
 or an element of **rtol** or **atol**  $<$  0.0,  
 or corresponding elements of **rtol** and **atol** are both 0.0,  
 or **neqn**  $\neq$  **npde**  $\times$  **npts** + **ncode**,  
 or **nxfix** not in the range 0 to **npts** – 2,  
 or fixed mesh point(s) do not coincide with any of the user-supplied mesh points,  
 or **dxmesh**  $<$  0.0,  
 or **ipminf**  $\neq$  0, 1 or 2,  
 or **xratio**  $\leq$  1.0,  
 or **con** not in the range  $0.1/(\mathbf{npts} - 1)$  to  $10/(\mathbf{npts} - 1)$ .

**ifail** = 2 (*warning*)

The underlying ODE solver cannot make any further progress, with the values of **atol** and **rtol**, across the integration range from the current point  $t = \mathbf{ts}$ . The components of **u** contain the computed values at the current point  $t = \mathbf{ts}$ .

**ifail** = 3 (*warning*)

In the underlying ODE solver, there were repeated error test failures on an attempted step, before completing the requested task, but the integration was successful as far as  $t = \mathbf{ts}$ . The problem may have a singularity, or the error requirement may be inappropriate.

**ifail** = 4

In setting up the ODE system, the internal initialization function was unable to initialize the derivative of the ODE system. This could be due to the fact that **ires** was repeatedly set to 3 in at least **pdedef**, **bnary** or **odedef**, when the residual in the underlying ODE solver was being evaluated.

**ifail** = 5

In solving the ODE system, a singular Jacobian has been encountered. You should check your problem formulation.

**ifail** = 6 (*warning*)

When evaluating the residual in solving the ODE system, **ires** was set to 2 in at least **pdedef**, **bnary** or **odedef**. Integration was successful as far as  $t = \mathbf{ts}$ .

**ifail** = 7

The values of **atol** and **rtol** are so small that the function is unable to start the integration in time.

**ifail** = 8

In one of **pdedef**, **bndary** or **odedef**, **ires** was set to an invalid value.

**ifail** = 9 (nag\_ode\_ivp\_stiff\_imp\_revcom (d02nn))

A serious error has occurred in an internal call to the specified function. Check the problem specification and all arguments and array dimensions. Setting **itrace** = 1 may provide more information. If the problem persists, contact NAG.

**ifail** = 10 (*warning*)

The required task has been completed, but it is estimated that a small change in **atol** and **rtol** is unlikely to produce any change in the computed solution. (Only applies when you are not operating in one step mode, that is when **itask**  $\neq$  2 or 5.)

**ifail** = 11

An error occurred during Jacobian formulation of the ODE system (a more detailed error description may be directed to the current error message unit). If using the sparse matrix algebra option, the values of **algotp**(29) and **algotp**(30) may be inappropriate.

**ifail** = 12

In solving the ODE system, the maximum number of steps specified in **algotp**(15) have been taken.

**ifail** = 13 (*warning*)

Some error weights  $w_i$  became zero during the time integration (see the description of **itol**). Pure relative error control (**atol**( $i$ ) = 0.0) was requested on a variable (the  $i$ th) which has become zero. The integration was successful as far as  $t = \mathbf{ts}$ .

**ifail** = 14

The flux function  $R_i$  was detected as depending on time derivatives, which is not permissible.

**ifail** = 15

When using the sparse option, the value of *lisave* or *lrsave* was not sufficient (more detailed information may be directed to the current error message unit).

**ifail** = 16

**remesh** has been changed between calls to nag\_pde\_1d\_parab\_remesh\_fd (d03pp), which is not permissible.

**ifail** = 17

The remeshing process has produced zero or negative mesh spacing. You are advised to check **monitf** and to try adjusting the values of **dxmesh**, **xratio** and **con**. Setting **ipminf** = 1 may provide more information.

**ifail** = -99

An unexpected error has been triggered by this routine. Please contact NAG.

**ifail** = -399

Your licence key may have expired or may not have been installed correctly.

**ifail** = -999

Dynamic memory allocation failed.

## 7 Accuracy

`nag_pde_1d_parab_remesh_fd` (d03pp) controls the accuracy of the integration in the time direction but not the accuracy of the approximation in space. The spatial accuracy depends on both the number of mesh points and on their distribution in space. In the time integration only the local error over a single step is controlled and so the accuracy over a number of steps cannot be guaranteed. You should therefore test the effect of varying the accuracy arguments, **atol** and **rtol**.

## 8 Further Comments

The argument specification allows you to include equations with only first-order derivatives in the space direction but there is no guarantee that the method of integration will be satisfactory for such systems. The position and nature of the boundary conditions in particular are critical in defining a stable problem. It may be advisable in such cases to reduce the whole system to first-order and to use the Keller box scheme function `nag_pde_1d_parab_remesh_keller` (d03pr).

The time taken depends on the complexity of the parabolic system, the accuracy requested, and the frequency of the mesh updates. For a given system with fixed accuracy and mesh-update frequency it is approximately proportional to **neqn**.

## 9 Example

This example uses Burgers Equation, a common test problem for remeshing algorithms, given by

$$\frac{\partial U}{\partial t} = -U \frac{\partial U}{\partial x} + E \frac{\partial^2 U}{\partial x^2},$$

for  $x \in [0, 1]$  and  $t \in [0, 1]$ , where  $E$  is a small constant.

The initial and boundary conditions are given by the exact solution

$$U(x, t) = \frac{0.1 \exp(-A) + 0.5 \exp(-B) + \exp(-C)}{\exp(-A) + \exp(-B) + \exp(-C)},$$

where

$$A = \frac{50}{E}(x - 0.5 + 4.95t),$$

$$B = \frac{250}{E}(x - 0.5 + 0.75t),$$

$$C = \frac{500}{E}(x - 0.375).$$

### 9.1 Program Text

```
function d03pp_example

fprintf('d03pp example results\n\n');

% Initialize input arguments
npde = nag_int(1);
m     = nag_int(0);
ts    = 0;
tout  = 0.2;
npts  = 61;
u     = zeros(npts, 1);
dx    = 1/(npts-1);
x     = [0:dx:1];
```

```

ncode = nag_int(0);
xi     = [];
rtol   = [5e-05];
atol   = [5e-05];
itol   = nag_int(1);
normt  = 'A';
laopt  = 'F';
algot  = zeros(30, 1);
remesh = true;
xfix   = [];
nrmesh = nag_int(3);
dxmesh = 0.5;
trmesh = 0;
ipminf = nag_int(0);
rsave  = zeros(5018, 1);
isave  = zeros(25, 1, nag_int_name);
itask  = nag_int(1);
itrace = nag_int(0);
ind     = nag_int(0);
cwsav  = {''; ''; ''; ''; ''; ''; ''; ''; ''};
lwsav  = false(100, 1);
iwsav  = zeros(505, 1, nag_int_name);
rwsav  = zeros(1100, 1);

% Integrate from t=0 to t=1 in steps of 0.1
usol = zeros(11,npts);
xsol = zeros(11,npts);
tsol = zeros(11,npts);
tout = [0.01,0.02,0.04,0.06,0.08,0.1,0.2,0.3,0.4,0.7,1.0];
for j = 1:11
    [ts, u, x, rsave, isave, ind, user, cwsav, lwsav, iwsav, rwsav, ifail] = ...
        d03pp( ...
            npde, m, ts, tout(j), @pdedef, @bdnary, @uvinit, ...
            u, x, ncode, @odedef, xi, rtol, atol, itol, normt, laopt, ...
            algot, remesh, xfix, nrmesh, dxmesh, trmesh, ipminf, @monitf, ...
            rsave, isave, itask, itrace, ind, cwsav, lwsav, iwsav, rwsav);

    xsol(j,:) = x;
    tsol(j,:) = ts;
    usol(j,:) = u;
end

fprintf(' Number of time steps           = %6d\n',isave(1));
fprintf(' Number of function evaluations = %6d\n',isave(2));
fprintf(' Number of Jacobian evaluations = %6d\n',isave(3));
fprintf(' Number of iterations           = %6d\n',isave(5));

fig1=figure;
mesh(xsol,tsol,usol);
title('Burger''s Equation solution using moving mesh');
xlabel('x');
ylabel('t');
zlabel('u(x,t)');
view(22,24);

function [p, q, r, ires, user] = pdedef(npde, t, x, u, ux, ncode, v, vdot, ...
                                       ires, user)

    p = zeros(npde, npde);
    q = zeros(npde, 1);
    r = zeros(npde, 1);

    p(1,1) = 1.0d0;
    r(1) = 0.005*ux(1);
    q(1) = u(1)*ux(1);

function [beta, gamma, ires, user] = bdnary(npde, t, u, ux, ncode, v, ...
                                           vdot, ibnd, ires, user)

    beta = zeros(npde, 1);
    gamma = zeros(npde, 1);

```

```

if (ibnd == 0)
    ue = exact(t,1,[0]);
else
    ue = exact(t,1,[1]);
end
gamma(1) = u(1) - ue(1);

function [u, v, user] = uvinit(npde, npts, nxi, x, xi, ncode, user)
    u = zeros(npde, npts);
    v = zeros(ncode, 1);

    u(1,1:npts) = exact(0,npts,x);

function [f, ires, user] = odedef(npde, t, ncode, v, vdot, nxi, xi, ...
                                ucp, ucpX, rcp, ucpt, ucptX, ires, user)
    f = zeros(ncode, 1);

function [fmon, user] = monitf(t, npts, npde, x, u, r, user)
    fmon = zeros(npts, 1);

    drdx = (r(2)-r(1))/(x(2)-x(1));
    fmon(1) = abs(drdx);
    for i = 2:npts-1
        % second derivative ..
        drdx = 2*(r(1,i+1)-r(1,i))/(x(i+1)-x(i-1));
        fmon(i) = abs(drdx);
    end
    fmon(npts) = fmon(npts-1);

function ue = exact(t,n,x);
    e = 0.005;
    ue = zeros(n,1);
    for i = 1:n
        a = (x(i)-0.25-0.75*t)/(4*e);
        b = (0.9*x(i)-0.325-0.495*t)/(2*e);
        if (a>0 && a>b)
            a = exp(-a);
            c = (0.8*x(i)-0.4-0.24*t)/(4*e);
            c = exp(c);
            ue(i) = (0.5+0.1*c+a)/(1+c+a);
        elseif (b>0 && b>=a)
            b = exp(-b);
            c = (-0.8*x(i)+0.4+0.24*t)/(4*e);
            c = exp(c);
            ue(i) = (0.1+c/2+b)/(1+c+b);
        else
            a = exp(a);
            b = exp(b);
            ue(i) = (1+a/2+0.1*b)/(1+a+b);
        end
    end
end

```

## 9.2 Program Results

d03pp example results

```

Number of time steps           =    199
Number of function evaluations =   4737
Number of Jacobian evaluations =     69
Number of iterations           =   505

```

