NAG Toolbox

nag_inteq_fredholm2_smooth (d05ab)

1 Purpose

nag_inteq_fredholm2_smooth (d05ab) solves any linear nonsingular Fredholm integral equation of the second kind with a smooth kernel.

2 Syntax

```
[f, c, ifail] = nag_inteq_fredholm2_smooth(k, g, lambda, a, b, odorev, ev, n)
[f, c, ifail] = d05ab(k, g, lambda, a, b, odorev, ev, n)
```

3 Description

nag_inteq_fredholm2_smooth (d05ab) uses the method of El–Gendi (1969) to solve an integral equation of the form

$$f(x) - \lambda \int_a^b k(x,s) f(s)\, ds = g(x)$$

for the function $f(x)$ in the range $a \leq x \leq b$.

An approximation to the solution $f(x)$ is found in the form of an n term Chebyshev series $\sum_{i=1}^{n}{}' c_i T_i(x)$, where $'$ indicates that the first term is halved in the sum. The coefficients c_i, for $i = 1, 2, \ldots, n$, of this series are determined directly from approximate values f_i, for $i = 1, 2, \ldots, n$, of the function $f(x)$ at the first n of a set of $m + 1$ Chebyshev points

$$x_i = \tfrac{1}{2}(a + b + (b - a) \times \cos[(i - 1) \times \pi/m]), \quad i = 1, 2, \ldots, m + 1.$$

The values f_i are obtained by solving a set of simultaneous linear algebraic equations formed by applying a quadrature formula (equivalent to the scheme of Clenshaw and Curtis (1960)) to the integral equation at each of the above points.

In general $m = n - 1$. However, advantage may be taken of any prior knowledge of the symmetry of $f(x)$. Thus if $f(x)$ is symmetric (i.e., even) about the mid-point of the range (a, b), it may be approximated by an even Chebyshev series with $m = 2n - 1$. Similarly, if $f(x)$ is anti-symmetric (i.e., odd) about the mid-point of the range of integration, it may be approximated by an odd Chebyshev series with $m = 2n$.

4 References

Clenshaw C W and Curtis A R (1960) A method for numerical integration on an automatic computer *Numer. Math.* **2** 197–205

El–Gendi S E (1969) Chebyshev solution of differential, integral and integro-differential equations *Comput. J.* **12** 282–287

5 Parameters

5.1 Compulsory Input Parameters

1: **k** – REAL (KIND=nag_wp) FUNCTION, supplied by the user.

k must compute the value of the kernel $k(x,s)$ of the integral equation over the square $a \leq x \leq b$, $a \leq s \leq b$.

```
[result] = k(x, s)
```

Input Parameters

1: **x** – REAL (KIND=nag_wp)
2: **s** – REAL (KIND=nag_wp)

The values of x and s at which $k(x, s)$ is to be calculated.

Output Parameters

1: **result**

The value of the kernel $k(x, s)$ evaluated at **x** and **s**.

2: **g** – REAL (KIND=nag_wp) FUNCTION, supplied by the user.

g must compute the value of the function $g(x)$ of the integral equation in the interval $a \leq x \leq b$.

```
[result] = g(x)
```

Input Parameters

1: **x** – REAL (KIND=nag_wp)

The value of x at which $g(x)$ is to be calculated.

Output Parameters

1: **result**

The value of $g(x)$ evaluated at **x**.

3: **lambda** – REAL (KIND=nag_wp)

The value of the parameter λ of the integral equation.

4: **a** – REAL (KIND=nag_wp)

a, the lower limit of integration.

5: **b** – REAL (KIND=nag_wp)

b, the upper limit of integration.

Constraint: $\mathbf{b} > \mathbf{a}$.

6: **odorev** – LOGICAL

Indicates whether it is known that the solution $f(x)$ is odd or even about the mid-point of the range of integration. If **odorev** is *true* then an odd or even solution is sought depending upon the value of **ev**.

7: **ev** – LOGICAL

Is ignored if **odorev** is *false*. Otherwise, if **ev** is *true*, an even solution is sought, whilst if **ev** is *false*, an odd solution is sought.

8: **n** – INTEGER

The number of terms in the Chebyshev series which approximates the solution $f(x)$.

Constraint: $\mathbf{n} \geq 1$.

5.2 Optional Input Parameters

None.

5.3 Output Parameters

1: **f**(**n**) – REAL (KIND=nag_wp) array

The approximate values f_i, for $i = 1, 2, \ldots, \mathbf{n}$, of the function $f(x)$ at the first **n** of $m + 1$ Chebyshev points (see Section 3), where

$m = 2\mathbf{n} - 1$ if **odorev** $= true$ and **ev** $= true$.

$m = 2\mathbf{n}$ if **odorev** $= true$ and **ev** $= false$.

$m = \mathbf{n} - 1$ if **odorev** $= false$.

2: **c**(**n**) – REAL (KIND=nag_wp) array

The coefficients c_i, for $i = 1, 2, \ldots, \mathbf{n}$, of the Chebyshev series approximation to $f(x)$. When **odorev** is *true*, this series contains polynomials of even order only or of odd order only, according to **ev** being *true* or *false* respectively.

3: **ifail** – INTEGER

ifail $= 0$ unless the function detects an error (see Section 5).

6 Error Indicators and Warnings

Errors or warnings detected by the function:

ifail $= 1$

On entry, **a** $\geq$ **b** or **n** < 1.

ifail $= 2$

A failure has occurred due to proximity to an eigenvalue. In general, if **lambda** is near an eigenvalue of the integral equation, the corresponding matrix will be nearly singular. In the special case, $m = 1$, the matrix reduces to a zero-valued number.

ifail $= -99$

An unexpected error has been triggered by this routine. Please contact NAG.

ifail $= -399$

Your licence key may have expired or may not have been installed correctly.

ifail $= -999$

Dynamic memory allocation failed.

7 Accuracy

No explicit error estimate is provided by the function but it is possible to obtain a good indication of the accuracy of the solution either

(i) by examining the size of the later Chebyshev coefficients c_i, or

(ii) by comparing the coefficients c_i or the function values f_i for two or more values of **n**.

8 Further Comments

The time taken by nag_inteq_fredholm2_smooth (d05ab) depends upon the value of **n** and upon the complexity of the kernel function $k(x, s)$.

9 Example

This example solves Love's equation:

$$f(x) + \frac{1}{\pi}\int_{-1}^{1} \frac{f(s)}{1 + (x - s)^2}\, ds = 1.$$

It will solve the slightly more general equation:

$$f(x) - \lambda \int_a^b k(x, s) f(s)\, ds = 1$$

where $k(x, s) = \alpha / \left(\alpha^2 + (x - s)^2\right)$. The values $\lambda = -1/\pi, a = -1, b = 1, \alpha = 1$ are used below.

It is evident from the symmetry of the given equation that $f(x)$ is an even function. Advantage is taken of this fact both in the application of nag_inteq_fredholm2_smooth (d05ab), to obtain the $f_i \simeq f(x_i)$ and the c_i, and in subsequent applications of nag_sum_chebyshev (c06dc) to obtain $f(x)$ at selected points.

The program runs for $\mathbf{n} = 5$ and $\mathbf{n} = 10$.

9.1 Program Text

```
     function d05ab_example

fprintf('d05ab example results\n\n');

k = @(x, s) 1/(1+(x-s)*(x-s));
g = @(x) 1;
lambda = -0.3183;
a = -1;
b = 1;
odorev = true;
ev     = true;
xval = [0:0.25:1];
ss = nag_int(2);

for n = 5:5:10;
  in = nag_int(n);
  [f, c, ifail] = d05ab(k, g, lambda, a, b, odorev, ev, in);
  fprintf('\nResults for N = %2d\n\n', n);
  fprintf('Solution and coefficients on first %2d Chebyshev points\n',n);
  fprintf('  i           x             f(i)         c(i)\n');
  cheb(1:n,1) = cos(pi*(1:n)/(2*n-1));
  fprintf('%3d%15.5f%15.5f%15.5e\n', [[1:n]' cheb f c]');

  [chebr, ifail] = c06dc(xval, a, b, c, ss);

  fprintf('\nSolution on evenly spaced grid\n');
  fprintf('     x           f(x)\n');
  fprintf('%8.4f%15.5f\n',[xval' chebr]')

end
```

9.2 Program Results

```
     d05ab example results

Results for N =  5

Solution and coefficients on first  5 Chebyshev points
  i           x            f(i)         c(i)
  1       0.93969       0.75572    1.41520e+00
  2       0.76604       0.74534    4.93840e-02
  3       0.50000       0.71729   -1.04758e-03
  4       0.17365       0.68319   -2.32817e-04
  5      -0.17365       0.66051    2.08903e-05

Solution on evenly spaced grid
     x           f(x)
  0.0000       0.65742
  0.2500       0.66383
  0.5000       0.68319
  0.7500       0.71489
  1.0000       0.75572

Results for N = 10

Solution and coefficients on first 10 Chebyshev points
  i           x            f(i)         c(i)
  1       0.98636       0.75572    1.41520e+00
  2       0.94582       0.75336    4.93840e-02
  3       0.87947       0.74639   -1.04751e-03
  4       0.78914       0.73525   -2.32749e-04
  5       0.67728       0.72081    1.99856e-05
  6       0.54695       0.70452    9.86754e-07
  7       0.40170       0.68825   -2.37956e-07
  8       0.24549       0.67404    1.85810e-09
  9       0.08258       0.66361    2.44829e-09
 10      -0.08258       0.65812   -1.65268e-10

Solution on evenly spaced grid
     x           f(x)
  0.0000       0.65742
  0.2500       0.66384
  0.5000       0.68319
  0.7500       0.71489
  1.0000       0.75572
```