Chapter 12
Ordinary Differential Equations

1 Scope of the Chapter

This chapter provides procedures for the numerical solution of systems of ordinary differential equations.

2 Available Modules

Module 12.1: `nag_ivp_ode_rk` — **Runge–Kutta methods for initial value problems**

This module contains a set of procedures for solving the *initial-value problem* for a system of first-order ordinary differential equations. The procedures are based on *Runge–Kutta* methods. Facilities are provided for:

- integrating across a given interval;
- integrating one step at a time;
- interpolating the computed solution;
- approximating the global error;
- returning performance statistics regarding the integration.

3 Background

3.1 Problem Formulation

For the module in this chapter the system of ordinary differential equations must be written in the form

$$\begin{aligned} y_1' &= f_1(t, y_1, y_2, \ldots, y_n) \\ y_2' &= f_2(t, y_1, y_2, \ldots, y_n) \\ &\vdots \\ y_n' &= f_n(t, y_1, y_2, \ldots, y_n), \end{aligned} \quad (1)$$

that is the system must be given in *first-order* form. The n dependent variables (the solution) $y_1, y_2, ..., y_n$ are functions of the independent variable t, and the differential equations give expressions for the derivatives $y_i' = dy_i/dt$ in terms of t and $y_1, y_2, \ldots, y_n$. For a system of n first-order ordinary differential equations, n associated boundary conditions are usually required to define the solution.

Reduction of higher-order systems to first-order form

A more general system may contain derivatives of higher order, but such systems can almost always be reduced to the first-order form (1) by introducing new variables. For example, taking the third-order equation

$$z''' + zz'' + k(l - z'^2) = 0$$

and writing $y_1 = z$, $y_2 = z'$ and $y_3 = z''$ we can obtain the first-order system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= -y_1 y_3 - k(l - y_2^2). \end{aligned} \quad (2)$$

3.2 Boundary Conditions

Consider the system (2). In this case $n = 3$ and we therefore require three boundary conditions in order to define the solution. These conditions must specify values of the dependent variables at certain points. For example, we have an *initial value problem* if the conditions are

$$\begin{aligned} y_1 &= 0.0 \quad \text{at} \quad t = 0.0 \\ y_2 &= 0.0 \quad \text{at} \quad t = 0.0 \\ y_3 &= 0.1 \quad \text{at} \quad t = 0.0. \end{aligned}$$

These conditions would enable us to integrate the equations numerically from the point $t = 0.0$ to some specified end-point. We have a *boundary-value problem* if the conditions are

$$\begin{aligned} y_1 &= 0.0 \quad \text{at} \quad t = 0.0 \\ y_2 &= 0.0 \quad \text{at} \quad t = 0.0 \\ y_3 &= 1.0 \quad \text{at} \quad t = 1.0. \end{aligned}$$

These conditions would be sufficient to define a solution on the range [0,1], but the problem cannot be solved by direct integration.

3.3 Stiff Systems

A special class of initial value problems are those for which the solutions contain rapidly decaying transient terms. Such problems are called *stiff* and require special methods for efficient numerical solution. Methods designed for non-stiff problems when applied to stiff problems tend to be very slow because they need small step-lengths to avoid numerical instability.

Currently this chapter only provides a module for non-stiff initial value problems.