NAG Library Function Document

nag_jacobian_theta (s21ccc)

1 Purpose

nag_jacobian_theta (s21ccc) returns the value of one of the Jacobian theta functions \( \theta_0(x, q), \theta_1(x, q), \theta_2(x, q), \theta_3(x, q) \) or \( \theta_4(x, q) \) for a real argument \( x \) and non-negative \( q < 1 \).

2 Specification

```c
#include <nag.h>
#include <nags.h>
double nag_jacobian_theta (Integer k, double x, double q, NagError *fail)
```

3 Description

nag_jacobian_theta (s21ccc) evaluates an approximation to the Jacobian theta functions \( \theta_0(x, q), \theta_1(x, q), \theta_2(x, q), \theta_3(x, q) \) and \( \theta_4(x, q) \) given by

\[
\begin{align*}
\theta_0(x, q) & = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^n \cos(2n\pi x), \\
\theta_1(x, q) & = 2 \sum_{n=0}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} \sin((2n+1)\pi x), \\
\theta_2(x, q) & = 2 \sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2} \cos((2n+1)\pi x), \\
\theta_3(x, q) & = 1 + 2 \sum_{n=1}^{\infty} q^n \cos(2n\pi x), \\
\theta_4(x, q) & = \theta_0(x, q),
\end{align*}
\]

where \( x \) and \( q \) (the nome) are real with \( 0 \leq q < 1 \).

These functions are important in practice because every one of the Jacobian elliptic functions (see nag_jacobian_elliptic (s21cbc)) can be expressed as the ratio of two Jacobian theta functions (see Whittaker and Watson (1990)). There is also a bewildering variety of notations used in the literature to define them. Some authors (e.g., Section 16.27 of Abramowitz and Stegun (1972)) define the argument in the trigonometric terms to be \( x \) instead of \( \pi x \). This can often lead to confusion, so great care must therefore be exercised when consulting the literature. Further details (including various relations and identities) can be found in the references.

nag_jacobian_theta (s21ccc) is based on a truncated series approach. If \( t \) differs from \( x \) or \(-x\) by an integer when \( 0 \leq t \leq \frac{1}{2} \), it follows from the periodicity and symmetry properties of the functions that \( \theta_1(x, q) = \pm \theta_1(t, q) \) and \( \theta_3(x, q) = \pm \theta_3(t, q) \). In a region for which the approximation is sufficiently accurate, \( \theta_1 \) is set equal to the first term (\( n = 0 \)) of the transformed series

\[
\theta_1(t, q) = 2 \sqrt{\frac{\lambda}{\pi}} e^{-\lambda t} \sum_{n=0}^{\infty} (-1)^n e^{-\lambda(n+\frac{1}{2})^2} \sinh((2n+1)\lambda t)
\]

and \( \theta_3 \) is set equal to the first two terms (i.e., \( n \leq 1 \)) of
\[ \theta_3(t, q) = \sqrt{\frac{2}{\pi}} e^{-\lambda^2 t} \left\{ 1 + 2 \sum_{n=1}^{\infty} e^{-\lambda^2 n^2 \cosh(2n\lambda t)} \right\}, \]

where \( \lambda = \frac{\pi^2}{\log q} \). Otherwise, the trigonometric series for \( \theta_1(t, q) \) and \( \theta_3(t, q) \) are used. For all values of \( x \), \( \theta_0 \) and \( \theta_2 \) are computed from the relations \( \theta_0(x, q) = \theta_3(\frac{1}{2} - |x|, q) \) and \( \theta_2(x, q) = \theta_1(\frac{1}{2} - |x|, q) \).

### 4 References


Tolke F (1966) *Praktische Funktionenlehre (Bd. II)* 1–38 Springer–Verlag


### 5 Arguments

1. \( k \) – Integer
   
   *Input*

   *On entry:* denotes the function \( \theta_k(x, q) \) to be evaluated. Note that \( k = 4 \) is equivalent to \( k = 0 \).

   *Constraint:* \( 0 \leq k \leq 4 \).

2. \( x \) – double
   
   *Input*

   *On entry:* the argument \( x \) of the function.

3. \( q \) – double
   
   *Input*

   *On entry:* the argument \( q \) of the function.

   *Constraint:* \( 0.0 \leq q < 1.0 \).

4. \( \text{fail} \) – NagError *
   
   *Input/Output*

   The NAG error argument (see Section 3.6 in the Essential Introduction).

### 6 Error Indicators and Warnings

**NE_ALLOC_FAIL**

Dynamic memory allocation failed.

See Section 3.2.1.2 in the Essential Introduction for further information.

**NE_INT**

On entry, \( k = \langle \text{value} \rangle \).

*Constraint:* \( k \leq 4 \).

On entry, \( k = \langle \text{value} \rangle \).

*Constraint:* \( k \geq 0 \).
NE_INTERNAL_ERROR

An internal error has occurred in this function. Check the function call and any array sizes. If the call is correct then please contact NAG for assistance.

An unexpected error has been triggered by this function. Please contact NAG. See Section 3.6.6 in the Essential Introduction for further information.

NE_NO_LICENCE

Your licence key may have expired or may not have been installed correctly. See Section 3.6.5 in the Essential Introduction for further information.

NE_REAL

On entry, q = \(\text{value}\).
Constraint: \(q < 1.0\).

On entry, q = \(\text{value}\).
Constraint: \(q \geq 0.0\).

7 Accuracy

In principle the function is capable of achieving full relative precision in the computed values. However, the accuracy obtainable in practice depends on the accuracy of the standard elementary functions such as \(\sin\) and \(\cos\).

8 Parallelism and Performance

Not applicable.

9 Further Comments

None.

10 Example

This example evaluates \(\theta_2(x, q)\) at \(x = 0.7\) when \(q = 0.4\), and prints the results.

10.1 Program Text

/* nag_jacobian_theta (s21ccc) Example Program. */
/* Copyright 2014 Numerical Algorithms Group. */
/* NAG C Library */
/* Mark 6, 2000. */
*/
#include <stdio.h>
#include <nag.h>
#include <nag_stdlib.h>
#include <nags.h>

int main(void)
{
    Integer exit_status = 0, k;
    NagError fail;
    double q, x, y;

    INIT_FAIL(fail);
    /* Skip heading in data file */
```c
#ifdef _WIN32
    scanf_s("%*[\n] ");
#else
    scanf("%*[\n] ");
#endif

printf("nag_jacobian_theta (s21ccc) Example Program Results\n");
printf(" k x q y\n");
#ifdef _WIN32
    while (scanf_s("%NAG_IFMT%lf%lf%*[\n]", &k, &x, &q) != EOF)
#else
    while (scanf("%NAG_IFMT%lf%lf%*[\n]", &k, &x, &q) != EOF)
#endif
{
    /* nag_jacobian_theta (s21ccc).
    * Jacobian theta functions with real arguments
    */
    y = nag_jacobian_theta(k, x, q, &fail);
    if (fail.code != NE_NOERROR)
    {
        printf("Error from nag_jacobian_theta (s21ccc).\n%s\n", fail.message);
        exit_status = 1;
        goto END;
    }
    printf("%2"NAG_IFMT" %4.1f %4.1f %13.4e\n", k, x, q, y);
}
END:
return exit_status;
}

10.2 Program Data

nag_jacobian_theta (s21ccc) Example Program Data
2 0.7 0.4 : Values of k, x and q

10.3 Program Results

nag_jacobian_theta (s21ccc) Example Program Results
k x q y
2 0.7 0.4 -6.9289e-01
```