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Structured Nearest Correlation Matrix Problems

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Joint work with
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Numerical Linear Algebra in Finance

- **Correlation matrices: nearness problems.**
- **Roots of transition matrices, $A^{1/p}$.**

Questions From Finance Practitioners

"Given a real symmetric matrix A which is almost a correlation matrix what is the best approximating (in Frobenius norm?) correlation matrix?"

"I am researching ways to make our company's correlation matrix positive semi-definite."

"Currently, I am trying to implement some real options multivariate models in a simulation framework. Therefore, I estimate correlation matrices from inconsistent data set which eventually are non psd."

Correlation Matrix

An $n \times n$ symmetric positive semidefinite matrix A with $a_{ii} \equiv 1$.

Properties:

- symmetric,
- 1s on the diagonal,
- eigenvalues nonnegative,
- off-diagonal elements between -1 and 1.

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Is this a correlation matrix?

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

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Spectrum: -0.4142, 1.0000, 2.4142.

Stock Research

- Sample correlation matrices constructed from vectors of stock returns.
- Can compute sample correlations of pairs of stocks based on days on which both stocks have data available.
- Resulting matrix of correlations is **approximate**, since built from inconsistent data sets.
- Relatively few vectors of observations available, so approximate correlation matrix has **low rank**.

How to Proceed

- $\surd$ Plug the gaps in the missing data, then compute an exact correlation matrix.
- $\times$ Make ad hoc modifications to matrix: e.g., shift negative e'vals up to zero then diagonally scale.
- $\surd$ Compute the **nearest correlation matrix**.

Problem

Compute distance

$$\gamma(A) = \min\{ \|A - X\| : X \text{ is a correlation matrix} \}$$

and a matrix achieving the distance.

Use a weighted Frobenius norm:

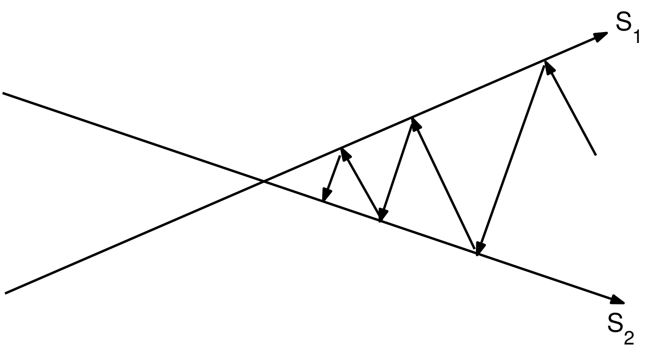
- $\|A\|_W = \|W^{1/2}AW^{1/2}\|_F$ (W pos def),
- $\|A\|_H = \|H \circ A\|_F$ ($h_{ij} > 0$),

where $\|A\|_F^2 = \sum_{i,j} a_{ij}^2$.

★ Constraint set is a closed, convex set, so unique minimizer.

Alternating Projections

von Neumann (1933), for subspaces.



Dykstra (1983) incorporated corrections for closed convex sets.

Projections

For $W \equiv I$.

- For $A = Q\,\mathrm{diag}(\lambda_i)Q^T$ let

$$\mathbf{P_S}(A) := Q\,\mathrm{diag}(\max(\lambda_i, 0))Q^T.$$

- $\mathbf{P_U}(A)$: replace diagonal by 1s.

More complicated for general W; see H (2002).

Algorithm (H, 2002)

Given symmetric $A \in \mathbb{R}^{n\times n}$ this algorithm computes nearest correlation matrix:

$\Delta S_0 = 0,\ Y_0 = A$
for $k = 1, 2, \dots$
  $R_k = Y_{k-1} - \Delta S_{k-1}$ % Dykstra's correction.
  $X_k = \mathbf{P_S}(R_k)$
  $\Delta S_k = X_k - R_k$
  $Y_k = \mathbf{P_U}(X_k)$
end

- X_k and Y_k both converge to solution.
- $O(n^3)$ operations per step.
- Linear convergence.
- Can add further constraints/projections (e.g., Toeplitz).

Newton Method

Qi & Sun (2006): convergent Newton method based on theory of **strongly semismooth matrix functions**.

Denote by A_+ projection of A onto psd matrices.

- Applies Newton to **dual** of $\min \frac{1}{2}\|A - X\|_F^2$ problem:
  $\min_{y \in \mathbb{R}^n} \frac{1}{2}\|(A + \mathrm{diag}(y))_+\|_F^2 - e^T y.$
- Dual problem is ctsly differentiable, but *not twice differentiable* $\Rightarrow$ use generalized Jacobian of gradient.
- **Globally** and **quadratically** convergent.
- Promising test results in Qi & Sun (2006).

The Dual Problem

$$\min_{y\in\mathbb{R}^n} \theta(y) := \frac{1}{2}\|(A+\mathrm{diag}(y))_+\|_F^2 - e^T y.$$

Solution of original problem is $X_* = (A+\mathrm{diag}(y_*))_+$.

Gradient:

$$\nabla\theta(y) = \mathrm{diag}(A+\mathrm{diag}(y))_+ - e.$$

Representative of **generalized Jacobian** of $\nabla\theta(y)$:

$$V_y h = \mathrm{diag}\left(P_y(W_y \circ (P_y^T H P_y))P_y^T\right),$$

where $H = \mathrm{diag}(h)$,

$$A + \mathrm{diag}(y) = P_y \mathrm{diag}(\lambda(y)) P_y^T$$

and

$$W_y = \begin{bmatrix} E_{11} & E_{12} & W_{13} \\ E_{12}^T & 0 & 0 \\ W_{13}^T & 0 & 0 \end{bmatrix}, \qquad E = ee^T.$$

Linear Systems in Newton Method

Newton equation is $V_k d_k = -\nabla\theta(y_k)$ with V_k pos semidef.
Alg requires us to compute d_k s.t.
$\|V_k d_k + \nabla\theta(y_k)\|_2 \le \eta_k \|\nabla\theta(y_k)\|_2$.

- Qi & Sun (2006) used **CG**.
- We use **minres**, which in solving $Ax = b$ **minimizes** $\|Ax_k - b\|_2$ on each iteration, producing monotonically decreasing residuals. Also more suited to semidefinite A than CG.

Hessian V_k

$$V_k h = \text{diag}\left(P_k(W_k \circ (P_k^T H P_k))P_k^T\right), \quad H = \text{diag}(h).$$

- V_k is always positive semidefinite.
- V_k can have "any" spectrum.
- Computing V_k requires $O(n^4)$ flops.

★ For ith column take $H = e_i e_i^T$, $h = e_i$.

★ Get diagonal via $e_i^T(V_k e_i)$.

Preconditioner

Can get **diagonal (Jacobi) preconditioner** in $O(n^3)$ flops:

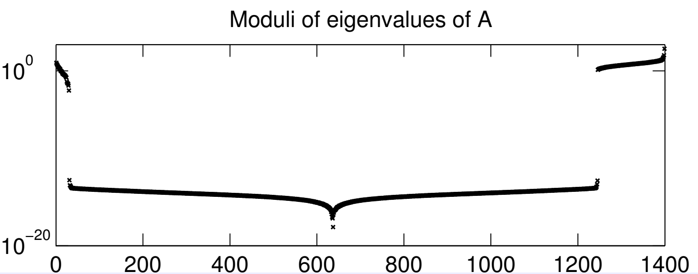
$$\begin{aligned} v_{ii} &= e_i^T P_k (W_k \circ P_k^T e_i e_i^T P_k) P_k^T e_i \\ &= p_i^T (W_k \circ p_i p_i^T) p_i \\ &= p_i^T \mathrm{diag}(p_i) W_k \mathrm{diag}(p_i) p_i \\ &= q_i^T W_k q_i, \end{aligned}$$

where $q_i \in \mathbb{R}^n$ with $q_i = p_i \circ p_i$. Thus

$$\begin{aligned} Q_k &= P_k \circ P_k && n^2 \text{ flops}, \\ M_k &= W_k [q_1 q_2 \dots q_n] = W_k Q_k && \le 2n^3 \text{ flops}, \\ v_{ii} &= q_i^T m_i, \quad i = 1\colon n && 2n^2 \text{ flops}. \end{aligned}$$

Numerical Example from Finance, $n = 1399$

$a_{ii} \equiv 1$, $|a_{ij}| \leq 1$, but not psd. $-8.5 \leq \lambda_i(A) \leq 339$.
A highly rank deficient with 1245 nonpositive ei'vals $\Rightarrow$ $\mathrm{rank}(X) \leq 154$.



$\|A - X_*\|_F = 20.96$.

Numerical Experiment

Stop when $\nabla\theta(y_k) \leq 10^{-7}n$.

cor1399, no precond

	T_{tot}	T_{mvp}	T_{eig}	Iters	# mvp
CG	213	146	63	7	42
Minres	171	104	62	7	30

cor1399, precond

	T_{tot}	T_{mvp}	T_{eig}	Iters	# mvp	T_{pre}
CG	142	77	53	6	22	9
Minres	111	45	53	6	13	9
Alt Proj	529			62		

Other Issues

- Choice of eigensolver: **divide and conquer** is *twice as fast* as **QR** and as **dqdr** on the cor1399 example (NAG MATLAB Toolbox).
- Armijo backtracking rule can be sensitive to rounding errors.
- Computed matrix does not have unit diagonal.
  Solution: set $X \leftarrow D^{-1/2} X D^{-1/2}$ where $D = \mathrm{diag}(X)$.

New Alg Versus Old

tol $= 10^{-7} n$.
Two 387×387 matrices from RiskMetrics.

	nearcor_new		nearcor		Altern. proj.	
	Time	Iter.	Time	Iter.	Time	Iter
cor1399	97	5	378	5	529	62
cor3120	814	4	5256	4	–	–
Risk-daily	0.39	0	0.47	0	1.02	2
Risk-monthly	0.36	0	0.53	0	1.22	2

- New alg is **G02AAF** in NAG Library Mark 22.

Conclusions

- ★ Feasible to compute **nearest** correlation matrix for problems for which spectral decomposition can be done.
- ★ Alternating projections
  - easy to implement,
  - can exploit low rank solutions,
  - linearly convergent,
- ★ Newton method faster and now have robust implementation. Will appear in NAG Library.
- ★ Theory and algorithms for structured problems under development.

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