Joint work with Jacques du Toit.

The views expressed in this presentation are the personal views of the speaker and do not necessarily reflect the views or policies of current or previous employers. Not guaranteed fit for any purpose. Use at your own risk.

*Chatham House Rules* apply to any reporting of presentation contents or comments by the speaker.
Before we start ...

If pricing uses first order sensitivities ...

... then hedging needs second order sensitivities
If **pricing** uses **second order** sensitivities ...

... then **hedging** needs **third order** sensitivities
Mathematically, there are theorems based on infinitesimals, and on finite differences

Infinitesimals (Taylor’s Theorem)
- Symbolic derivatives of compact equations using analytic expressions (analytic expression have arbitrary operators)
- Symbolic derivatives of extended equations (i.e. code, computers have only +, −, ×, / operators, roughly speaking)

Large overlap between these

Finite differences (Newton’s Theorem)
- Small, e.g. for sensis
- Large, e.g. for stresses

Financially we are interested in effects of market movements
Definition (Taylor’s Theorem)

Let \( k \in \mathbb{N} > 0 \) and \( f : \mathbb{R} \rightarrow \mathbb{R} \) be \( k \) times differentiable at \( a \in \mathbb{R} \) then \( \exists h_k(x) : \mathbb{R} \rightarrow \mathbb{R} \) s.t.

\[
f(x) = P_k(x) + h_k(x)(x - a)^k
\]

where \( P_k(x) \) is the \( k \)-th order Taylor polynomial

\[
P_k(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \ldots + \frac{f^{(k)}(a)}{k!}(x - a)^k
\]

and \( \lim_{x \to a} h_k(x) = 0 \)

Hence we can define a remainder

\[
R_k(x) := f(x) - P_k(x) = o(|x - a|^k), \quad x \to a
\]

and if \( f \) is \( k + 1 \) times differentiable on the open interval with \( f^{(k)} \) continuous on the closed interval between \( a \) and \( x \) then, by the Mean Value Theorem,

\[
R_k(x) = \frac{f^{(k+1)}(v)}{(k + 1)!}(x - a)^{k+1}, \quad v \in [a, x]
\]
Assuming everything works, then with first order derivatives

\[ R_1(x) = \frac{f''(v)}{2} (x - a)^2, \quad v \in [a, x] \]

and second

\[ R_2(x) = \frac{f'''(v)}{6} (x - a)^3, \quad v \in [a, x] \]

cannot do better than this

Finite market moves make the above optimistic
Mathematical Limitations

- Require
  - \( f \) must be analytic (i.e. Taylor series must converge to \( f \))
  - Non-analytic example where Taylor coefficients are all zero at zero
    \[
    f(x) = \begin{cases} 
      e^{-1/x^2} & x > 0 \\ 
      0 & x \leq 0 
    \end{cases}
    \]
  - \( k \) times differentiable at \( a \)
  - \( k + 1 \) times differentiable on the open interval with \( f^{(k)} \) continuous on the closed interval between \( a \) and \( x \)
  - Must be able to get the derivatives
  - There is a radius of convergence within which the approximation works
Financial Limitations

- Exercise boundaries limit availability of first-order differentiability
- Trade life-cycle: fixings (e.g. with averaging instruments); resets; coupons; notional payments; maturity; transformation (e.g. swaption to swap)
- Opaque/illiquid model parameters
- Self and Counterparty life-cycle: rating transitions; default; regulatory permissions
- Self-Counterparty: CSA change; SwapAgent®, i.e. CTM-to-STM; collateral change
- Calibration instrument life-cycle: Futures rolls; index rolls; CDS rolls
- Significant market dates: FOMC meetings; Central Bank meetings
- Information releases: inflation publication; employment; etc.
- Gap events: currency life-cycle (start, end, division = pegs); regulatory changes
Assume all life-cycle (trade, entities, calibration instrument) and market dates are already included in the explain.

First order: all

Cross-gamma: highly dependent on correlations
- base-base, e.g. IR curvature
- base-base, e.g. IR-FX, IR-CM
- base-vol, e.g. IR and IR vol
- vol-vol, e.g. FX smile flattening

Diagonal-gamma used but less commonly (also depends on definition of diagonal- vs cross-)
How good are your correlations?

- Market-implied correlations similar to other market-implied items (rates, vols) but generally require taking positions in several instruments to hedge.
- Historical correlations change as slowly as the calibration algorithm.
  - Few models for stochastic correlation — part of more general Wrong Way Risk problem.
  - Confidence interval width feeds into Prudential Valuation capital.
- Default correlation is challenging to estimate from market or historical data.
General purpose efficient approach in (Kenyon and Green 2015)

- $t = 0$ CVA, FVA hedging needs
  - Forward derivatives of portfolio
  - Jacobian chain back to calibration instruments
  - Cross-gamma of CR-XX vital to capture market risk

- Forward derivatives
  - First order: SIMM; CCP IM; FRTB; FRTB-CVA
  - Second order: FRTB (approximate curvature)

- Accuracy requirements?
  - Hedging
  - Compression
  - Incremental trading
  - Allocation
MVA: first-order sensis in pricing

- SIMM, (ISDA-SIMM-2 2017)
  - Delta-vega approach, i.e. first-order
  - Many papers and presentations on using forward derivatives to calculate SIMM
  - Hedging SIMM requires second-order sensis

- Regulatory and CCP methods
  - Generally, historical VaR or Expected Shortfall approach (or moving to this)
  - Direct approach (Green and Kenyon 2015) — main issue is change in the key scenarios (Kenyon and Green 2015; Andreasen 2017), which is a jump risk
  - If approximate CCP IM re-using forward sensis developed for SIMM (suggestion from (Chan 2017)) then need second order sensis

- Hedging effects of IM (regulatory or CCP) on option exercise also required (Green and Kenyon 2017), also needs second order derivatives

- No Market Risk capital on MVA (or FVA)
### KVA, FRTB-CVA-SA: first-order sensis in pricing

<table>
<thead>
<tr>
<th></th>
<th>IR, INF</th>
<th>FX</th>
<th>Credit (Cpty)</th>
<th>Credit (Exp)</th>
<th>Equity</th>
<th>Commodities</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Risk Factors</strong></td>
<td>Delta, Vega</td>
<td>Delta, Vega</td>
<td>Delta</td>
<td>Delta, Vega</td>
<td>Delta, Vega</td>
<td>Delta, Vega</td>
</tr>
<tr>
<td><strong>Risk Buckets</strong></td>
<td>Currency (not dom)</td>
<td>Currency</td>
<td>Sectors (e.g. IG)</td>
<td>Sectors (e.g. IG)</td>
<td>Sectors (large cap)</td>
<td>Group</td>
</tr>
<tr>
<td><strong>Delta</strong></td>
<td>Main IR 3 pieces; INF and other IR 1</td>
<td>FX spot</td>
<td>5 pieces</td>
<td>Single per bucket</td>
<td>Single per bucket</td>
<td>Single per bucket</td>
</tr>
<tr>
<td>Method</td>
<td>Relative</td>
<td>Relative</td>
<td>Absolute</td>
<td>Absolute</td>
<td>Relative</td>
<td>Relative</td>
</tr>
<tr>
<td><strong>Vega</strong></td>
<td>Single</td>
<td>Single</td>
<td>NA</td>
<td>Single per bucket</td>
<td>Single per bucket</td>
<td>Single per bucket</td>
</tr>
<tr>
<td>Method</td>
<td>Relative</td>
<td>Relative</td>
<td>NA</td>
<td>Relative</td>
<td>Relative</td>
<td>Relative</td>
</tr>
</tbody>
</table>

- KVA using FRTB-CVA-SA requires first order forward sensitivities of CVA
- Hedging KVA on FRTB-CVA-SA requires second order

(c) C.Kenyon 2017
FRTB, KVA: second-order sensis in pricing?

- Is this relevant? Main issue is dealing with future trading to maintain $t = 0$ Market Risk Capital level.

- **FRTB-IMA**
  - Expected Shortfall(97.5%), 10-day plus liquidity modification, calibrated to a period of stress
  - Non-modellable risk factors (NMRF)
  - Default risk charge (DRC)

- **FRTB-SA**
  - Sensitivity based: as FRTB-CVA but more detailed + curvature
  - Default risk charge
  - Residual risk add-on

- Some work on KVA pricing (Andreasen 2017), but not hedging or allocation. Generally follow pattern of (Kenyon and Green 2015)

- One open question is whether suggestion of (Chan 2017) to re-use sensitivities obtained for MVA/SIMM for FRTB-IMA is workable
Conclusions

- Many limitations on practical application of Taylor’s Theorem in financial markets from non-differentiability requiring smoothing — and error bound requires next order so full revaluation often more practical
- First-order sensis in pricing so second-order for hedging:
  - MVA: SIMM; possibly CCP approximation
  - KVA: FRTB-CVA; FRTB-SA (if bump for curvature)
- Second-order sensis in pricing so third-order for hedging
  - KVA: FRTB-SA (if use for curvature)
- Other second-order sensi uses
  - PnL explain
  - FRTB PnL explain
- Revaluation required:
  - Stress testing
  - FRTB-IMA at $t = 0$
- Unclear whether VaR/ES at $t = 0$ will be permitted using sensis (delta-gamma-vega) rather than full revaluation going forward


