A Finite Volume Scheme for Calibrating Stochastic Local Volatility Models

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Stochastic Local Volatility Model
We consider models of the form
\[\begin{align*}
    dX_t &= \left(r_d - r_f - \frac{\sigma_{SLV}^2}{2} \right) dt + \sigma_{SLV}(X_t, \tau) \psi(V_t) dt + \sigma_{SLV}(X_t, \tau) \psi(V_t) dW_t^{(1)} \\
    dV_t &= \kappa(\eta - V_t) dt + \xi \sqrt{V_t} dW_t^{(2)}
\end{align*}\]

Results are presented for the popular SLV Heston model ($\alpha = 1/2$) but the method handles other values of $\alpha$ as well.

Calibrating the Leverage Surface
We must compute a leverage surface $\sigma_{SLV}$ consistent with today’s prices. This is done by providing a local volatility surface $\sigma_{LV}$ and then using Gyöngy’s Theorem
\[\sigma_{SLV}^2(x, \tau) = \frac{2}{\tau} \int_0^\infty \bar{p}(x, v, \tau; x_0, v_0) dv\]
where $\bar{p}$ is the transition density of the process $(X_t, V_t)_{t \geq 0}$. This density can be obtained by solving the Kolmogorov forward equation
\[\frac{\partial}{\partial \tau} p + \frac{\partial}{\partial v} \left(2\sigma_{LV}^2(v)p \right) + \frac{\partial^2}{\partial v^2} \left(\eta - v \right) p = \frac{\partial}{\partial v} \left(\frac{\xi}{\sigma_{LV}} \frac{\partial}{\partial v} p \right) - \frac{\partial}{\partial \tau} \left(\kappa \eta - \sigma_{LV}^2 \right) p\]

A Second Order ADI Finite Volume Scheme
Solving the forward equation with finite differences is difficult: there are no known boundary conditions at $v = 0$. We apply a standard second order finite volume discretisation to the un–transformed forward equation and impose zero–flux conditions at the boundaries. When $\alpha = 1/2$ we use upwinding at $v = 0$. We smooth the Dirac delta initial condition with 4 Rannacher half steps, whereafter we use Hundsdorfer-Verwer ADI as the time stepping method. We iterate Gyöngy’s Theorem twice at each time step: this gives sufficient convergence for $\sigma_{SLV}$.

Validation Against Heston Model
We took $\alpha = 1/2$ and validated the finite volume method against the known Heston transition density. Convergence and density plots are shown below. We used $m_1$ mesh points in the $x$ direction and $m_2 = m_1/2$ points in the $v$ direction. Convergence is second order when the Feller condition is satisfied, but drops to between first and second order when the Feller condition is violated due to the first order upwinding.

SLV Calibration on EUR/USD Data from 2 March 2016
We took market FX quotes and chose challenging stochastic parameters
\[\alpha = 0.5 \quad \kappa = 0.3 \quad \eta = 0.04 \quad \sigma = 0.61 \quad \rho = 0.63 \quad T = 0.5\]
The Feller value is 0.65 indicating strong violation of the Feller condition. A modified SSVI method gave the input local volatility surface:

We calibrated the model using the finite volume ADI method and obtained the following leverage surface:

To evaluate the accuracy of the calibration, we compared the marginal SLV density in $x$ with the density of the local volatility model at $T$. The two agree to within 1e-3. We then compared the implied volatility quotes under the local volatility model to the implied volatilities under the calibrated SLV model. The difference $\varepsilon = |\sigma_{imp, LV} - \sigma_{imp, SLV}|$ is given below.

<table>
<thead>
<tr>
<th>$K/S_0$</th>
<th>0.75</th>
<th>0.80</th>
<th>0.90</th>
<th>1.0</th>
<th>1.10</th>
<th>1.20</th>
<th>1.25</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{imp, LV}$</td>
<td>20.48</td>
<td>19.10</td>
<td>16.16</td>
<td>12.50</td>
<td>11.52</td>
<td>11.93</td>
<td>12.30</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>1.3e-2</td>
<td>8.9e-3</td>
<td>2.9e-3</td>
<td>7.4e-4</td>
<td>4.3e-3</td>
<td>1.0e-2</td>
<td>1.5e-2</td>
</tr>
</tbody>
</table>

Implementation and Performance
The Fortran code uses preconditioned GMRES to solve the Rannacher systems. The independent tridiagonal ADI systems are all solved in parallel. On a 12 core, dual socket machine we observe the following performance for calibrating an SLV model on a 600 x 300 grid:

<table>
<thead>
<tr>
<th>Num Threads</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>24</th>
<th>48</th>
</tr>
</thead>
<tbody>
<tr>
<td>All Rannacher steps</td>
<td>75ms</td>
<td>59ms</td>
<td>37ms</td>
<td>24ms</td>
<td>18ms</td>
<td>13ms</td>
<td>7ms</td>
</tr>
<tr>
<td>One ADI step</td>
<td>12.6s</td>
<td>8.6s</td>
<td>5.6s</td>
<td>3.6s</td>
<td>2.6s</td>
<td>1.1s</td>
<td>0.78s</td>
</tr>
</tbody>
</table>

Code Availability
The code will be in the forthcoming version of the NAG Library. Please contact support@nag.co.uk for early evaluation options. Further performance and algorithmic improvements are currently being explored.

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