Software Issues in Wavelet Analysis of Financial Data

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Results Matter. Trust NAG.
Overview

- Why Use Wavelets?
- Wavelet transforms
- Multi-Resolution Analysis
- Software implementations and algorithms
- Choosing a wavelet method
- Some Applications
Why Use Wavelets?

Signal → stream of data in time

To analyse structure of time series:

- **DFT** (Discrete Fourier Transform) → frequency representation
- **STFT** (Short Time Fourier Transform, Gabor) → uses a time window to give localisation in time – imposes a scale which leads to aliasing of components
- **Wavelet Transform** → shifted and scaled basis functions allow localisation in time and frequency
- **Uncertainty principle**: cannot achieve simultaneous time and frequency resolution
Wavelet Transforms
Decompose time series, $x(t)$, by convolution with dilated and translated mother wavelet, $\psi(t)$.

- **Continuous (CWT)**

  \[
  d(u, s) = \int_{-\infty}^{\infty} x(t) \psi_{u,s}(t) \, dt,
  \]

  \[
  \psi_{u,s}(t) = \frac{1}{\sqrt{s}} \psi \left( \frac{t - u}{s} \right)
  \]

  e.g. Morlet wavelet,

  \[
  \psi(t) = \frac{1}{\sqrt{2\pi}} e^{-ikt} e^{-t^2/2}
  \]

- **Discrete (DWT)**

  Filter pair: $G$ – high pass
  $H$ – low pass
  with $D$ – down-sampling

  \[
  (Hx)_k = \sum_n h_{n-k} x_n
  \]
Wavelet Transforms (2)

- CWT requires:

\[ \int_{-\infty}^{\infty} \psi(t) dt = 0 \quad \int_{-\infty}^{\infty} |\psi(t)|^2 \, dt = 1 \]

- DWT (orthogonal filter pair) requires:

\[ \sum_{n} h_n h_{n+2j} = 0, \quad \sum_{n} h_n^2 = 1, \quad g_n = (-1)^n h_{1-n} \]

\[ \sum_{n} g_n g_{n+2j} = 0, \quad \sum_{n} g_n^2 = 1, \quad \sum_{n} h_n g_{n+2j} = 0 \]
Wavelet Functions

Morlet (k=5)

Gaussian 3rd derivative

Daubechies:

DB4

DB12

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Relation between *Continuous* and *Discrete* Transforms

- Scaling function is fixed point of $H$:
  \[
  \varphi(t) = \sqrt{2} \sum_j h_j \varphi(2t - j)
  \]

- Mother wavelet:
  \[
  \psi(t) = \sqrt{2} \sum_j g_j \varphi(2t - j)
  \]

- Time series:
  (see Strang & Nguyen, 1997)
  \[
  x(t) = \sum_k c^J_k \varphi_J(t - 2^{-J} k)
  \]

- CWT detail coefficients
  \[
  x(t) = \sum_k c^R_k \varphi_R(t - 2^{-R} k) + \sum_{j=R}^{J-1} d^j_k \psi_j(t - 2^{-j} k)
  \]

- Wavelet coefficients
  \[
  d^j_k = \int \psi_j(t - 2^{-j} k) x(t) dt
  \]

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Stationary DWT (SDWT)

- Note: DWT is **NOT** translation invariant
  choosing **odd** entries in series, in place of **even** ones when down-sampling gives a different orthogonal transformation
- DWT MRA – *choice of shift at each level gives multiple possible sets of coefficients*
- SDWT – **NO** down-sampling,
  *pad filters with zeros in MRA*
  *includes all DWT MRA possibilities*
  *translation invariant, but increases storage*
  *can relate wavelet coefficients to data*
Translation invariant SDWT

\[ x(t) \]

\[ d_1 \]

\[ d_2 \]

\[ d_3 \]

\[ d_4 \]

\[ s_4 \]
Choosing a Wavelet Method

- **CWT**
  *Continuous transform*
  *Visualise as surface*

- **DWT/MRA**
  *Discrete, multiresolution*
  *Efficient storage of signal*

- **Matching Pursuit**
  *Adapt basis to data*
  *At each level of MRA choose waveform to minimise residual passed to next level*

- **SDWT**
  *Translation invariant*
  *No down-sampling*
CWT: how can quantitative information be obtained?
CWT (2)

- Find common normalisation for wavelet spectrum – ensure wavelet has unit energy at each scale
- Choice of wavelet:
  - non-orthogonal is useful for time series, but highly redundant
- Choice of scales:
  - can use arbitrary set of scales to show structure
- Cone of influence:
  - for finite length series defines where edge effects occur
- Relate to Fourier frequency

E.g. see Torrence and Compo (1998)
Software implementation and algorithms

- Reproducibility is desirable –

- Edge effects –
  *contaminate ends of transform for finite signals* – various end conditions used to reduce their effect: periodic extension, reflection, zero-padding …

- CWT –
  *implement quadrature and convolution in time domain or else convolution with Fourier Transform of wavelet in frequency domain*

- Parallel implementation
Implementation and algorithms (2)

- Definition of forward and inverse transforms –
  \[ \text{DWT orthogonality conditions allow for different choices of forward and inverse transforms} \]
- Pre-processing of data –
  \[ \text{data may need cleaning, interpolation to produce homogeneous series, \ldots} \]
Applications

- De-noising
- Identifying seasonality
- Self-similarity
- Prediction
- Estimation of variance
De-Noising

- Transform data into wavelet domain
- Apply thresholding – suppress *smallest* coefficients
- Transform back

**Use:**
DWT for efficient storage  –  SDWT to align with data
De-noised data can help modelling of underlying structure
e.g. Capobianco (1997) – analysis of Nikkei index
Donoho and Johnstone (1998)
Identifying Seasonality

- Apply SDWT to data
- Wavelet detail coefficients at a given level capture a particular range of frequencies
- Identify detail coefficients carrying seasonal periodicity
- Filter out seasonal effects

e.g. Gencay et al. (2001) for application to FX returns
Self-similarity

- Scalogram represents energy of series in the wavelet coefficients
  e.g. Jamdee and Los (2004) use Morlet wavelet CWT for analysis of interest rate series and calculation of Hurst exponent

Prediction

- Apply backward looking wavelet MRA to time series
- Use wavelet coefficients as input to a neural network model for prediction
  e.g. Renaud et al. (2004)
Estimation of Variance

For DWT coefficients $w$
SDWT coefficients $w^s$

$$\| x \|^2 = \| w \|^2 = \| w^s \|^2$$

Since,

$$\| x \|^2 \propto \text{Var}(x)$$

The wavelet transform provides an alternative representation of the variance (Percival, 1995)
Summary

1) Wavelet transforms provide a rigorous framework for data analysis in time and frequency
2) Software implementations vary in their choices of transform definitions
3) Wavelet analysis provides an important tool for determining the structure of time series arising in finance