Portfolio Optimization using the NAG Library

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Abstract

NAG Libraries have many powerful and reliable optimizers which can be used to solve large portfolio optimization and selection problems in the financial industry. Below is an introduction into the notation and techniques used in portfolio optimization. We discuss some sample problems and present help in choosing an appropriate NAG optimizer. Finally, there is a section on handling transaction cost for the portfolio optimization.

1 Introduction

The selection of assets or equities is not just a problem of finding attractive investments. Designing the correct portfolio of assets cannot be done by human intuition alone and requires the use of numerical optimization techniques. The Numerical Algorithms Group Ltd (NAG) is world renowned for its work on numerical algorithms, and NAG routines for optimization are being used extensively in industry, commerce and academia. Many leading financial companies and institutions employ NAG optimizers to select, diversify and rebalance their portfolios. They are also used by business and management schools for teaching and research.

Any investor would like to have the highest return possible from an investment. However, this has to be counterbalanced by the amount of risk the investor is able or desires to take. The expected return and the risk as measured by the variance (or the standard deviation, which is the square-root of the variance) are the two main characteristics of a portfolio. Unfortunately, equities with high returns usually also have high risk.

The performance of a portfolio can be quite different from the performance of individual components of the portfolio. The risk of a properly constructed portfolio from equities in leading markets could be half the sum of the risks of individual assets in the portfolio. This is due to complex correlation patterns between individual assets or equities. A good optimizer can exploit the correlations, the expected returns, the risk (variance) and user constraints to obtain an optimized portfolio. NAG optimization routines can deliver optimized and diversified portfolios to match investor expectations.

The mathematical problem of portfolio optimization was initiated by Professor Harry Markowitz in the fifties and he was rewarded with a Nobel Prize in Economics in 1990 which he shared with Professors William Sharpe and Merton Miller [10]. NAG optimizers can handle the classical Markowitz optimization problems [9], [11], [12] and many modern day extensions [5], [15], [19], [20], [21]. NAG also provides a consultancy service to the financial sector to solve mathematical, numerical, programming problems associated with portfolio optimization, automatic differentiation, bond and option pricing, and other areas.

Portfolio optimization is often called mean-variance (MV) optimization. The term 'mean' refers to the mean or the expected return of the investment and the 'variance' is the measure of the risk associated with the portfolio. The mathematical problem can be formulated in many ways but the principal problems can be summarized as follows:

1. Minimize risk for a specified expected return
2. Maximize the expected return for a specified risk
3. Minimize the risk and maximize the expected return using a specified risk aversion factor
4. Minimize the risk regardless of the expected return
5. Maximize the expected return regardless of the risk

The above problems could have linear, nonlinear, equality or inequality constraints. The first three problems are essentially mathematically equivalent. The fourth problem gives minimum variance solutions which are for cautious investors. It is also used for comparison and benchmarking of other portfolios. The fifth problem gives the upper bound of the expected return which can be attained; this is also useful for comparisons.

When market conditions (for example expected returns or correlations between assets) or the investor's risk preferences change, it is advisable to rebalance the portfolio. Any of the above problems can be solved relative to an existing portfolio or a benchmark, with the idea of matching or exceeding the benchmark performance. Solutions to the above problems are called

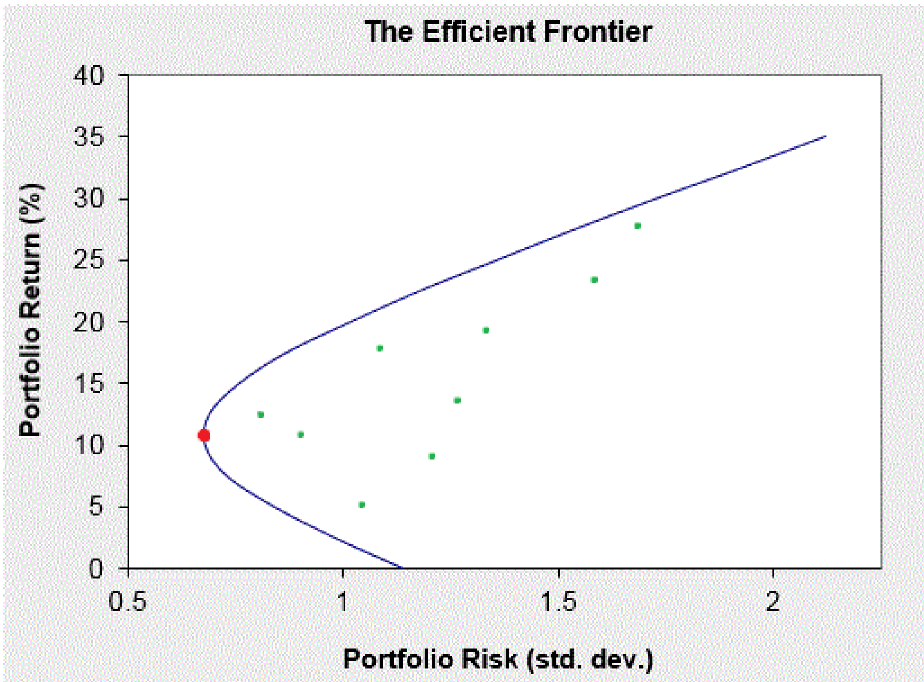

Figure 1: The curve describes the efficient frontier of maximum and minimum return for a given risk. All realized portfolios lie to its right.

mean-variance (MV) efficient. The efficient points in the Return-Risk graph are called the Efficient Frontier, as shown in Figure 1.

The transaction costs associated with purchasing a new portfolio or rebalancing a portfolio could represent a significant amount to the investor. NAG optimization routines can handle transactions costs and they may significantly affect the composition of the portfolio.

2 Notation

We use notation common in portfolio optimization:

x the vector of portfolio weights

μ the vector of expected returns[1]

[1] Care should be taken when performing MV optimization with linear vs. compound returns [14].

Σ the covariance matrix (usually computed from historical data)

l_i the lower bound for the i^{th} asset

u_i the upper bound for the i^{th} asset

Typical problems have the weights summing to unity (known as a fully invested constraint):

$$\sum_{i=1}^{n} x_i = 1$$

and bounds on the variables:

$$l_i \leq x_i \leq u_u$$

It is also possible that one x_i correspond to a cash-equivalent. Note that if $x_i \leq 0$ this corresponds to short selling the asset.

In addition, investors may have linear constraints on certain groups of assets. They may wish to allocate a minimum of 50% of the portfolio in a subset of the equities. This can be formulated and input into the optimizer via the matrix A where:

$$L \leq Ax \leq U$$

Although the bounds on x_i could be included in the definition of general linear constraints, we prefer to distinguish between them for reasons of computational efficiency. For equality constraints, lower and upper limits are set to equal values. It is also possible to set the upper limits to ∞ and the lower limits to $-\infty$.

3 Example Optimization Problems

When formulating your model, there are numerous combinations of objective functions and constraints that the NAG optimizers can handle. Below we present some of the more common problems:

Minimize Risk (Markowitz Model) $\min_x x^T \Sigma x$

This is the classic portfolio optimization problem where the investor is looking to minimize the risk when setting a desired level of return:

$$\mu^T x = \mu^*$$

Constraints and bounds on variables may or may not be present.

Resampled MV Optimization $\frac{1}{m}\left(\sum_{j=1}^{m} \arg \min_{\mu_j^T x = \mu^*} x^T \Sigma_j x\right)$

Resampled mean variance is used in the presence of linear constraints and bound constraints only and is similar to the Markowitz model except the expected returns are assumed to be random variables. The idea is to simulate the returns μ_j and covariances Σ_j of m outcomes. For each of these scenarios, perform a portfolio optimization to find the optimal holdings and then average the results. This approach can be useful when there exists some prior knowledge of the distribution of assets [15], or when estimation from historical data is very difficult.

Maximize the Expected Returns $\max_x \mu^T x$

This is where the investor is looking to get the most returns without taking risk or other factors into account. Care should be taken with these optimizations as the solutions can often place the weights on a few assets regardless of risk.

Risk/Cost Aversions $\max_x \mu^T x - f(x)$

The cost aversion is a variant of Markowitz Portfolio Optimization. The goal is still to maximize the expected returns, but at the expense of other factors denoted by the penalty term f. Some common forms of the penalty are given below:

$f(x)$	Interpretation
$x^T \Sigma x$	Risk Aversion
$c^T \lvert \bar{x}_i - x_i \rvert$	Linear Trading Costs
$c_1^T \lvert \bar{x} - x \rvert + c_2^T \lvert \bar{x}_i - x_i \rvert^2$	Quadratic Trading Costs

An investor may choose a combination of the above forms or choose a non-linear function for f.

Here, $\bar{x}$ is the portfolio prior to reallocation and x is the new allocation. The case $\bar{x}_i = x_i$ is where no trading (and thus no cost) occurs for the ith asset. Note that the absolute value function for trading costs can be troublesome for optimizers. A way to handle such situations is detailed in Section 5.

Black-Litterman Model $$\max_x \left[x^T \bar{\mu} - \left(\frac{\delta}{2} \right) x^T \bar{\Sigma} x \right]$$

Published by Fischer Black and Robert Litterman in 1992, the Black-Litterman model provides a combination of the past performance of assets with future views on performance. A review of the basic formulas and notation for Black-Litterman are presented below. (For a detailed derivation see [7] or [13].)

Instead of the expected return vector μ, we have $\bar{\mu}$, a vector that combines the equilibrium risk premiums with prior views on the market. The distribution $\bar{\mu}$ can be calculated using Bayesian analysis.

$$\bar{\mu} = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\Pi + P'\Omega^{-1}Q]$$

with covariance matrix

$$\bar{M}^{-1} = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}$$

The variables above are:

$\Pi = \delta\Sigma w_{eq}$ is the equilibrium risk premiums

δ the risk aversion parameter

w_{eq} the market portfolio

$\bar{\Sigma} = \Sigma + \bar{M}^{-1}$ the updated covariance matrix

P the view matrix

Q returns on each view

τ the uncertainty of Q

$\Omega = diag(P(\tau\Sigma)P^T)$ the covariance matrix of views

To implement Black-Litterman Optimization, the investor begins by formulating views on the market as well as a confidence interval on them. This is then input into the above formulas for $\bar{\mu}$ and $\bar{\Sigma}$, which are used in the optimization. The selection of the risk aversion parameter δ is based on prior heuristics. This can either be set to a specific value or calibrated using past market data.

4 NAG Optimization

The objective function can take on many forms depending upon the problem and investor preferences. Common forms of the objective function for optimization problems are given below.

Objective Function	Problem Type
$c^T x$	Linear Programming(LP)
$c^T x + x^T \Sigma x$	Quadratic Programming(QP)
$c^T x + \frac{1}{2}\|\|b - \Sigma x\|\|^2$	Least Squares (LS)
$f(x)$	Nonlinear Programming (NLP)

Note that for the NLP problems, convexity may be an issue. The objective function may have many local extrema and the resulting numerical solution may not be the global optimum [18].

Once the model has been formulated, it is time to choose an optimization routine. Table 1 shows some of the types of problems the NAG Library can handle and offers a recommendation in selecting routines.

Note that some optimization functions can handle more than one type of problem. A nonlinear optimizer can be used on QP problems, for example, but it is computationally inefficient to do so.

4.1 Covariance/Correlation Matrix

When attempting to compute the covariance matrix Σ from past returns, rounding or incomplete data may make the computed matrix indefinite. Performing a computation with such a matrix may produce bizarre results

Objective Function	Routine Name	Constraints	Dense/Sparse
LP, Convex QP, & LS	nag_opt_lin_lsq	quadratic	dense
LP & QP	nag_opt_qp	quadratic	dense
LP & Convex QP	nag_opt_sparse_convex_qp	quadratic	sparse
NLP	nag_opt_nlp	nonlinear	dense
NLP	nag_opt_nlp_revcomm	nonlinear	dense
NLP	nag_opt_nlp_sparse	nonlinear	sparse

Table 1: NAG Optimization Routines

if the problem is not sufficiently constrained. Fortunately many of the NAG algorithms will detect an indefinite matrix. Should the computed matrix be indefinite, a Nearest Correlation Matrix (NCM) routine from Chapter G02 of the NAG Library might be useful. These functions will find a correlation matrix that is closest in some sense to the original computed matrix, and can incorporate weights/factor structures.

4.2 Forward and Direct Communication

Most of the optimization routines are based on forward communications. In such programs, the routine is called only once to obtain the results and the user supplies all the necessary information to the NAG routine via a subroutine. However, in some circumstances, it is necessary to do the optimization step by step and call the user routine repeatedly to get fresh information.

The NAG routine **nag_opt_nlp** is a forward communication routine and **nag_opt_nlp_revcomm** is the direct communication equivalent. This direct communication routine is particularly useful when it is called from another language (i.e., Microsoft VBA) where the callback functions required by a forward communication algorithm may be unwieldy to code.

4.3 Cold and Warm Starts

Cold starts refer to solutions of the problem from scratch. However, if the routines are called repeatedly then approximate solutions are available from previous solutions. In that case, the initial conditions for the next iteration may be supplied from the previous. Such warm start facilities are available for many NAG optimization routines.

4.4 Derivatives of Objective

NAG recommends the user supply as many derivatives as a particular algorithm can use for computational efficiency. In cases where the derivatives of your particular objective function are difficult to calculate or do not exist at certain points, the NAG routine will automatically calculate partial differentials for the supplied functions via finite differencing.

When finite differencing is too expensive or inaccurate, or the derivatives are very difficult to code, another technique that may be used is Algorithmic Differentiation (AD). NAG has worked very closely with RWTH Aachen University to deliver AD tools and solutions to customers worldwide [17].

4.5 Global Optimization

It may turn out that your objective function has many local minima, in addition to a global minimum. Such problems can be much harder to solve than local optimization problems because it is difficult to determine whether a potential minimum is global, and because of the nonlocal methods required to avoid becoming trapped near local optima. If this is the case then we recommend algorithms from Chapter E05 of the NAG Library which contains Global Optimization Methods.

5 Transaction Costs

In the classical work of Markowitz, transaction costs associated with buying and selling of equities are not considered. However, the importance of incorporating transaction costs in building portfolios and also in rebalancing existing portfolios are well recognized. In general, transaction costs are not trivial enough to be neglected and the optimal portfolio depends upon the total cost of transactions. Let us model the buying of additional quantities of asset i by:

$$p_i = \begin{cases} (x_i - \bar{x}_i) & \text{for } x_i > \bar{x}_i \\ 0 & \text{for } x_i \leq \bar{x}_i \end{cases}$$

where x_i is the new portfolio weight of equity i, $\bar{x}_i$ is the original weight of equity i. Similarly, we model the selling of asset i by:

$$q_i = \begin{cases} 0 & \text{for } x_i > \bar{x}_i \\ (\bar{x}_i - x_i) & \text{for } x_i \leq \bar{x}_i \end{cases}$$

Note that both p_i and q_i cannot be simultaneously non-zero since you do not wish to both buy and sell an asset at the same time.

Let $\phi(x)$ be the objective function for minimization without transaction costs. The new objective function with transaction costs is then given by:

$$\phi(x) + \sum_{i=1}^{n}(g_i p_i + h_i q_i)$$

where g_i and h_i are, respectively, the costs associated with buying and selling quantities of assets.

By including both p_i and q_i as additional (constrained) variables, the problem is rendered smooth. This does, however, double the number of problem variables, or triple them when g_i and h_i are distinct.

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