Confluent and Gauss Hypergeometric Functions ${}_1F_1(a;b;x)$ (S22BA, S22BB) and ${}_2F_1(a,b;c;x)$ (S22BE, S22BF)

New at mark 24 are routines for the evaluation of the Confluent hypergeometric function $M(a,b,x) = {}_1F_1(a;b;x)$, and the Gauss hypergeometric function $F(a,b,c,x) = {}_2F_1(a,b;c;x)$, with real valued parameters a, b and c, and real argument x. Two routines are available for each, the first returning the value of the functions M (S22BA) and F (S22BE) directly, and the others returning the solutions in the scaled forms $M = m_{fr} \times 2^{m_{sc}}$ (S22BB) and $F = f_{fr} \times 2^{f_{sc}}$ (S22BF)[1].

${}_1F_1(a,b;x)$ has a wide variety of applications, including CIR processes and pricing Asian options. ${}_2F_1(a,b;c;x)$ is also used in financial option pricing, and has a plethora of uses in general scientific computing. Furthermore, it is possible to express many special functions in terms of these hypergeometric functions, including the exponential function, the binomial series, Bessel functions, complete elliptic integrals, Chebyschev, Legendre, Laguerre and Gegenbauer polynomials and the incomplete gamma function.

$M(a,b,x)$ is a linearly independent solution to the differential equation,

$$x\frac{d^2M}{dx^2} + (b-x)\frac{dM}{dx} - aM, \tag{1}$$

and can be defined via the power series,

$$M(a,b,x) = {}_1F_1(a;b;x) = \sum_{j=0}^{\infty} \frac{(a)_j x^j}{(b)_j j!}. \tag{2}$$

[1]Routines for ${}_1F_1$ area available in all mark 24 libraries. Routines for ${}_2F_1$ are available in the mark 24 C library, and will be available in the mark 25 Fortran Library and toolbox for MATLAB®.

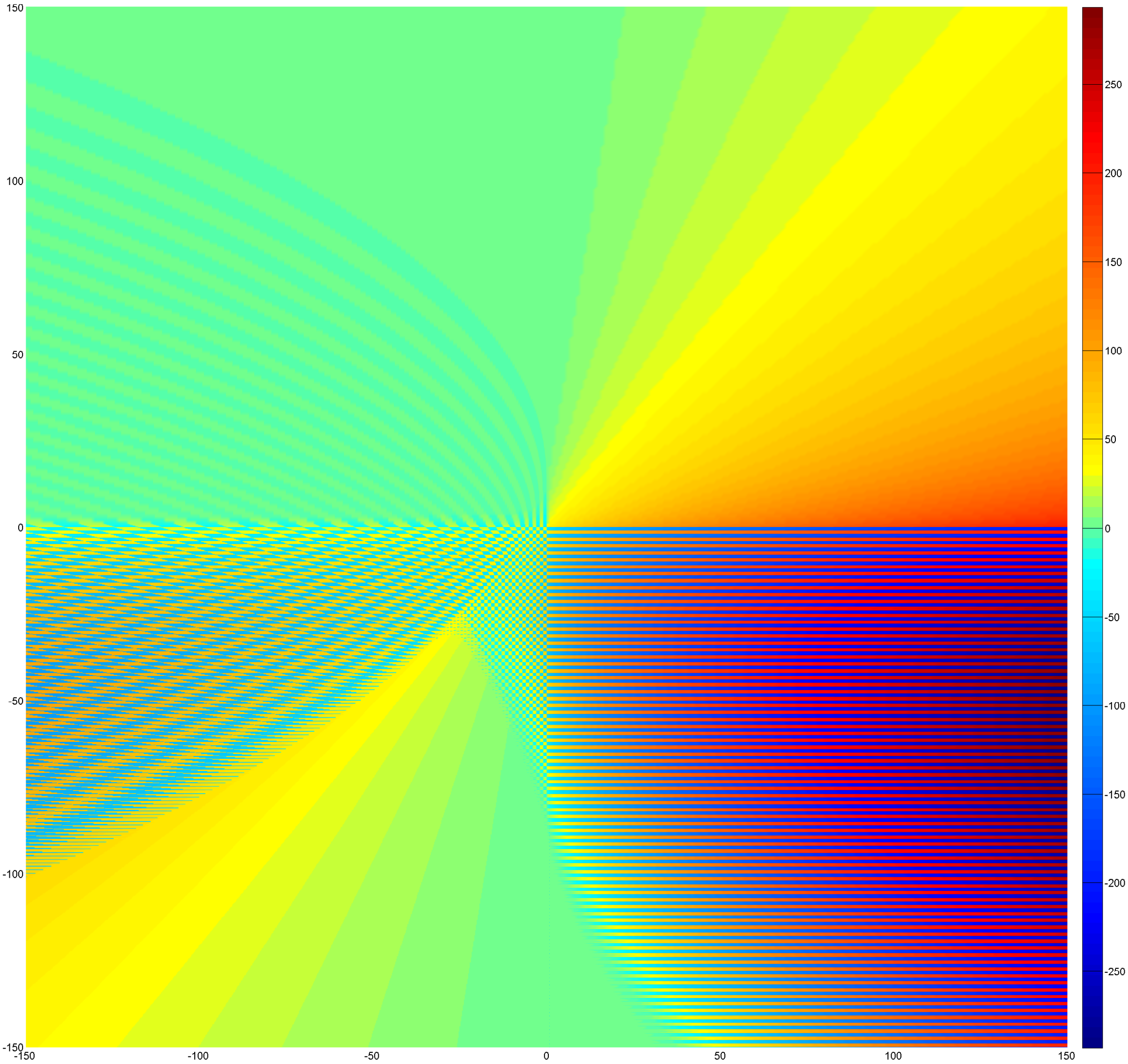

Figure 1: $M(a, b, x)$ for $a \in [-150, 150], b \in (-150, 150]$ and $x = 25$.

Here, $(\alpha)_j = 1(\alpha)(\alpha+1)\ldots(\alpha+j-1)$ is the rising Pockhammer function of α. The series in (2) is convergent for all a, b and x, and for real valued parameters and argument will always converge to a real value. However, M rapidly exceeds the double precision exponential limit for even moderate values of the parameters a, b and x ($\sim O(100)$), and as such the availability of the fractional component m_{fr} and scale m_{sc} from S22BB allows for meaningful results to be returned over much greater ranges. Figure (1) shows $M(-150 \leq a \leq 150, -150 < b < 150, x = 25)$, plotted as $\frac{M}{|M|}\log_2(|M|+1)$ to emphasize the highly oscillatory nature and scale of the function.

Similarly, $F(a, b, c, x)$ is a linearly independent solution to the differential equation,

$$x(1-x)\frac{d^2F}{dx^2} + (c-(a+b-1)x)\frac{dF}{dx} - abF, \tag{3}$$

and can be defined via the power series,

$$F(a,b,c,x) = {}_2F_1(a,b;c;x) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j x^j}{(c)_j j!}. \tag{4}$$

This is only strictly convergent when $|x| < 1$. For $x < 1$, real valued transformations exist mapping the argument to the interval $(0, 1)$, and hence for $x < 1$, F is real valued and finite. For $x = 1$, if $c > a + b$, $F = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$. Degenerate cases where $c - a$ or $c - b$ are negative integers may also be resolved. If $x = 1$ and $c \leq a + b$, F is infinite, and the sign of F may be determined as $x \to 1$ from below. Hence the magnitude of F may rapidly exceed the double precision exponent limit, particularly as $x \to 1$. For $x > 1$, F typically has a nonzero imaginary component. Figure (2) shows F with $a = -9.75$ for $x = 0.1$ and the cases where a linear transformation always results in a real value. These are plotted as $\frac{F}{|F|}\log_2(|F| + 1)$.

S22BB and S22BF also accept the parameters a, b and c as integral and decimal fractional components to increase the accuracy in the floating point calculations. This can provide a significant improvement to the solution when small perturbations to integral values are required. For example, consider the solutions for $M(-199.999999, -400.00001, 600)$. S22BA gives $M(a, b, x) = -0.1320802726327450 \times 10^{295}$, whereas S22BB gives $M(a_{ni} + a_{dr}, b_{ni} + b_{dr}, x) = -0.1320803101722191 \times 10^{295}$, where the nearest integer and decimal remainder components are $a_{ni} = -200, a_{dr} = 10^{-6}, b_{ni} = -200$ and $b_{dr} = -10^{-6}$. The relative differences is $\sim O(10^{-7})$.

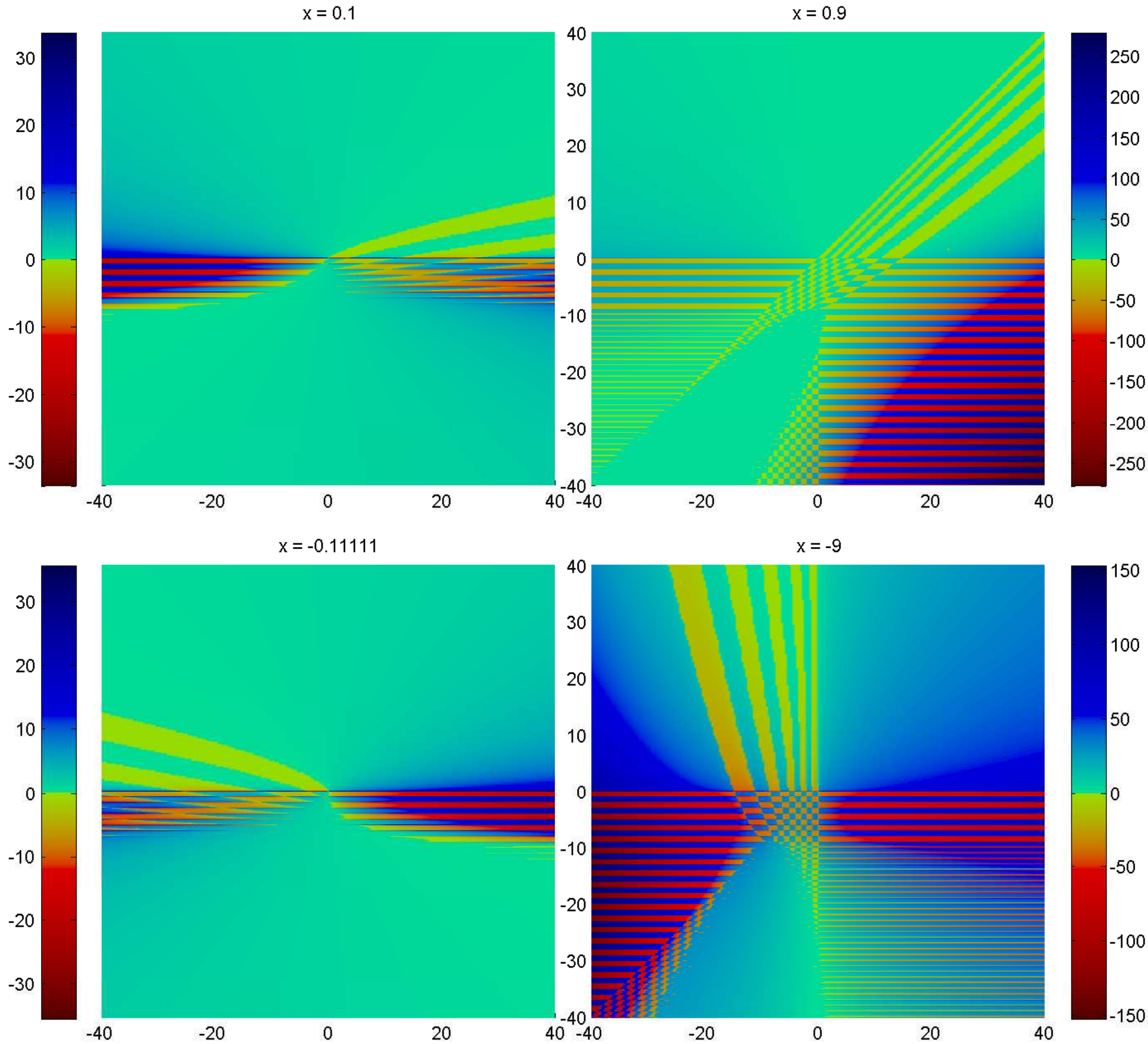

Figure 2: ${}_2F_1(a, b; c; x)$ for $a = -9.75, b \in [-40, 40], c \in (-40, 40]$. Clockwise from top left, $x = 0.1, 0.9, -9, -0.1111$.