NAG C Library Chapter Introduction

s – Approximations of Special Functions

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1 Scope of the Chapter

This chapter provides functions for computing some of the most commonly required special functions of mathematics. See Chapter g01 for probability distribution functions and their inverses.

2 Accuracy

Many of the functions in this chapter evaluate real-valued functions of a single real variable \( x \). The accuracy of the computed value is discussed in each function document, along the following lines.

Let \( \Delta \) be the absolute error in the argument \( x \), and \( \delta \) be the relative error in \( x \).

If we ignore errors that arise in the argument by propagation of data errors etc. and consider only those errors that result from the fact that a real number is being represented in the computer in floating-point form with finite precision, then \( \delta \) is bounded and this bound is independent of the magnitude of \( x \); for example, on an \( n \)-digit machine

\[
|\delta| \leq 10^{-n}.
\]

(This of course implies that the absolute error \( \Delta = x\delta \) is also bounded but the bound is now dependent on \( x \).)

Let \( E \) be the absolute error in the computed function value \( f(x) \), and \( \epsilon \) be the relative error. Then

\[
E \approx |f'(x)| \Delta E \approx |x f'(x)| \delta \epsilon \approx |x f'(x) / f(x)| \delta.
\]

If possible, the function documents discuss the last of these relations, that is the propagation of relative error, in terms of the error amplification factor \( |x f'(x) / f(x)| \). But in some cases, such as near zeros of the function which cannot be extracted explicitly, absolute error in the result is the quantity of significance and here the factor \( |x f'(x)| \) is described.

In general, testing of the functions has shown that the behaviour of the actual errors follows fairly well these theoretical relations. In regions where the error amplification factors are less than one, or of the order of one, the errors are slightly larger than the above predictions. The errors are here limited largely by the finite precision of arithmetic in the machine, but \( \epsilon \) is normally no more than a few times greater than the bound on \( \delta \). In regions where the amplification factors are large, of order of ten or greater, the theoretical analysis gives a good measure of the accuracy obtainable.

3 Approximations to Elliptic Integrals

3.1 Definitions of Symmetrised Elliptic Integrals

Four functions in the chapter compute symmetrised variants of the classical elliptic integrals. These alternative definitions have been suggested by Carlson and he also developed the basic algorithms used in this chapter.

Important Advice : users who encounter elliptic integrals in the course of their work are strongly recommended to look at transforming their analysis directly to one of the Carlson forms, rather than the traditional canonical Legendre forms. In general, the extra symmetry of the Carlson forms is likely to simplify the analysis, and these symmetric forms are much more stable to calculate. In case that is not possible, rules for computing the Legendre forms in terms of the Carlson forms are given below.

The symmetrised integral of the first kind, which is computed by nag_elliptic_integral_rf (s21bbc), is defined by

\[
R_F(x, y, z) = 12 \int_0^\infty \frac{dt}{\sqrt{(t + x)(t + y)(t + z)}},
\]

where \( x, y, z \geq 0 \) and at most one may be equal to zero. The normalisation factor, 12, is chosen so as to make

\[
R_F(x, x, x) = 1 / \sqrt{x}.
\]

If any two of the variables are equal, \( R_F \) degenerates into the function \( R_C \), which is computed by
nag_elliptic_integral_rc (s21bac):

\[ R_C(x, y) = R_F(x, y, y) = 12 \int_0^\infty \frac{dt}{\sqrt{t + x(t + y)}} \]

where the argument restrictions are now \( x \geq 0 \) and \( y \neq 0 \).

The symmetrised integral of the second kind, which is computed by nag_elliptic_integral_rd (s21bcc), is defined by

\[ R_D(x, y, z) = 32 \int_0^\infty \frac{dt}{\sqrt{(t + x)(t + y)(t + z)^3}} \]

with \( z > 0, \ x \geq 0 \) and \( y \geq 0 \) but only one of \( x \) or \( y \) may be zero.

This function is a degenerate special case of the integral of the third kind, which is computed by nag_elliptic_integral_rj (s21bdc) and is defined by

\[ R_J(x, y, z, \rho) = 32 \int_0^\infty \frac{dt}{\sqrt{(t + x)(t + y)(t + z)(t + \rho)}} \]

with \( \rho \neq 0, \ x, y, z \geq 0 \) with at most one equality holding. Thus \( R_D(x, y, z) = R_J(x, y, z, z) \). The normalisation of both these functions is chosen so that

\[ R_D(x, x, x) = R_J(x, x, x, x) = 1/(x\sqrt{x}) \].

### 3.2 Relationships with Classical Elliptic Integrals

The above forms can be related to the more traditional Legendre canonical forms as follows. Let

\[ q = \cos^2 \phi, \quad r = 1 - m \sin^2 \phi, \quad s = 1 + n \sin^2 \phi, \]

where \( 0 < \phi \leq 12\pi \). Then we have

the incomplete elliptic integral of the first kind:

\[ F(\phi|m) = \int_0^{\sin \phi} (1 - t^2)^{-1/2}(1 - mt^2)^{-1/2}dt = \sin \phi R_F(q, r, 1); \]

the incomplete elliptic integral of the second kind:

\[ E(\phi|m) \]

the elliptic integral of the third kind:

\[ \Pi(n; \phi|m) \]

and the complete elliptic integral of the second kind:

\[ E(m) = \int_0^{\pi/2} (1 - m \sin^2 \theta)^{1/2}d\theta = R_F(0, 1 - m, 1) - 13mR_D(0, 1 - m, 1). \]

The function \( R_C \) is related to the logarithm or inverse hyperbolic functions if \( 0 < y < x \), and to the inverse circular functions if \( 0 \leq x \leq y \). For example

\[ \ln x = (x - 1)R_C \left( \frac{1 + x}{2}, x \right), \quad x > 0; \]

\[ \arcsin x = xR_C(1 - x^2, 1), \quad |x| \leq 1; \]

\[ \arcsinh x = xR_C(1 + x^2, 1), \quad \text{etc.} \]

In general this method of calculating elementary functions is not recommended as there are usually much
more efficient specific functions available. However $R_C$ may be used, for example, to compute
$\ln x/(x - 1)$ when $x$ is close to 1, without the loss of significant figures that occurs when $\ln x$ and $x - 1$
are computed separately.

4 Available Functions

s10aac Hyperbolic tangent, $\tanh x$
s10abc Hyperbolic sine, $\sinh x$
s10acc Hyperbolic cosine, $\cosh x$
s11aac Inverse hyperbolic tangent, $\text{arctanh } x$
s11abc Inverse hyperbolic sine, $\text{arcsinh } x$
s11acc Inverse hyperbolic cosine, $\text{arccosh } x$
s13aac Exponential integral $E_I(x)$
s13acc Cosine integral $C_i(x)$
s13ade Sine integral $S_i(x)$
s14aac Gamma function $\Gamma(x)$
s14abc Log Gamma function $\ln(\Gamma(x))$
s14acc Gamma function $\Gamma(x)$
s14afc Derivative of the psi function $\psi(z)$
s14bac Incomplete gamma functions $P(a, x)$ and $Q(a, x)$
s15abc Cumulative normal distribution function, $P(x)$
s15acc Complement of cumulative normal distribution function, $Q(x)$
s15adc Complement of error function, $\text{erfc } x$
s15ace Error function, $\text{erf } x$
s17acc Bessel function $Y_0(x)$
s17ade Bessel function $Y_1(x)$
s17ace Bessel function $J_0(x)$
s17afe Bessel function $J_1(x)$
s17age Airy function $A_i(x)$
s17ahc Airy function $B_i(x)$
s17ajc Airy function $A'_i(x)$
s17akc Airy function $B'_i(x)$
s17alc Zeros of Bessel functions $J_\alpha(x)$, $J'_\alpha(x)$, $Y_\alpha(x)$ or $Y'_\alpha(x)$
s18acc Modified Bessel function $K_0(x)$
s18adc Modified Bessel function $K_1(x)$
s18ace Modified Bessel function $I_0(x)$
s18acf Modified Bessel function $I_1(x)$

s18ccd Scaled modified Bessel function $e^x K_0(x)$
s18cdd Scaled modified Bessel function $e^x K_1(x)$
s18ccc Scaled modified Bessel function $e^{-|z|} I_0(x)$

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s18cfc  Scaled modified Bessel function $e^{-|x|}I_1(x)$
s18ecc  Scaled modified Bessel function $e^{-x}I_{v/4}(x)$
s18edc  Scaled modified Bessel function $e^xK_{v/4}(x)$
s18ecc  Modified Bessel function $I_{v/4}(x)$
s18ecf  Modified Bessel function $K_{v/4}(x)$
s18egc  Modified Bessel functions $K_{\alpha+n}(x)$ for real $x > 0$, selected values of $\alpha \geq 0$ and $n = 0, 1, \ldots, N$
s18ehc  Scaled modified Bessel functions $e^xK_{\alpha+n}(x)$ for real $x > 0$, selected values of $\alpha \geq 0$ and $n = 0, 1, \ldots, N$
s18ejc  Modified Bessel functions $I_{n+1}(x)$ or $I_{n-1}(x)$ for real $x \neq 0$, non-negative $\alpha < 1$ and $n = 1, 2, \ldots, |N| + 1$
s18ekc  Bessel functions $J_{n+1}(x)$ or $J_{n-1}(x)$ for real $x \neq 0$, non-negative $\alpha < 1$ and $n = 1, 2, \ldots, |N| + 1$
s19aac  Kelvin function ber $x$
s19abc  Kelvin function bei $x$
s19acc  Kelvin function ker $x$
s19adc  Kelvin function kei $x$
s20acc  Fresnel integral $S(x)$
s20adc  Fresnel integral $C(x)$
s21bac  Degenerate symmetrised elliptic integral of 1st kind $R_C(x, y)$
s21bbc  Symmetrised elliptic integral of 1st kind $R_F(x, y, z)$
s21bdc  Symmetrised elliptic integral of 2nd kind $R_D(x, y, z)$
s21bdc  Symmetrised elliptic integral of 3rd kind $R_J(x, y, z, r)$
s21dac  Elliptic integrals of the second kind with complex arguments
s21cbc  Jacobian elliptic functions sn, cn and dn with complex arguments
s21ccc  Jacobian theta functions with real arguments
s22aac  Legendre and associated Legendre functions of the first kind with real arguments

See also Chapter g01 for probability distribution functions and their inverses.